Consider the Solow neoclassical one-sector growth model with rate of population growth \( n = .04 \), and aggregate production function

\[
F (K, L) = \frac{KL}{K + L}.
\]

(a) Find the intensive production function \( y = f(k) \), output per capita as a function of the capital/labor ratio \( k = K/L \). What is Solow’s equation for the evolution of the capital/labor ratio?
(b) If the saving fraction $s = .20$ and the initial capital/labor ratio is 1, describe the long-run behavior of the economy. What is consumption *per capita* in the steady state?

(c) What is meant by the golden rule of saving? For the steady state, calculate the real interest rate $r = f'(k)$, and compare this value to $n$. Relate the steady state to the golden rule.
Macroeconomics

Golden Rule—Example

Answer

(a)

\[ f(k) = \frac{F(K, L)}{L} = \frac{KL}{K+L} = \frac{K}{K+L} = \frac{L}{L} \frac{K}{L} + 1 = \frac{k}{k+1}. \]

More directly,

\[ f(k) = F(k, 1) = \frac{k}{k+1}. \]
Solow’s equation for the evolution of the capital/labor ratio is

$$\frac{dk}{dt} = sf(k) - nk.$$  

The change in capital $dk/dt$ (capital deepening *per capita*) is the difference between $sf(k)$ (saving *per capita*) and $nk$ (capital widening *per capita*).
(b) In the long run, the economy converges to steady-state growth. To solve for the steady-state capital/labor, set $dk/dt = 0$ and solve for $k$:

$$0 = sf(k) - nk$$

$$= s \frac{k}{k+1} - nk.$$  

Cancelling $k$ gives

$$0 = s \frac{1}{k+1} - n,$$

and solving gives

$$k = \frac{s}{n} - 1 = \frac{.20}{.04} - 1 = 4.$$
If initially $k = 1$, it will gradually increase until $k = 4$. In the long run, output per capita is $f(4) = \frac{4}{4+1} = .8$. Saving per capita is $sf(k) = .20 \times .8 = .16$, and consumption per capita is $.8 - .16 = .64$. Population, capital, and output all grow at rate $n = .04$. 
(c) The golden rule of saving is to pick $s$ to maximize steady state \textit{per capita} consumption.

The following three conditions are equivalent:

a) golden-rule saving rate;

b) population growth = interest rate;

c) saving rate = capital income share.
The real interest rate is the marginal product of capital, the slope of the intensive production function:

\[ r = f'(k) \]

\[ = \frac{1}{k+1} - \frac{k}{(k+1)^2} \]

\[ = \frac{k+1}{(k+1)^2} - \frac{k}{(k+1)^2} \]

\[ = \frac{1}{(k+1)^2} \]

\[ = \frac{1}{(4+1)^2} \]

\[ = .04. \]
Here we have the second condition, $n = r$, so $s = .20$ is the golden rule of saving.
Of course the third condition for the golden rule must also hold. Profit *per capita*

$$rk = .04 \times 4 = .16$$

equals saving *per capita*. 

The capital income share is profit *per capita* divided by income *per capita*,

$$\frac{.16}{.80} = .20,$$

which is the saving rate.