Question

Consider the Solow neoclassical one-sector growth model with rate of population growth $n = .04$, and aggregate production function

$$F(K,L) = \frac{KL}{K+L}.$$

(a) Find the intensive production function $y = f(k)$, output *per capita* as a function of the capital/labor ratio $k = K/L$. What is Solow's equation for the evolution of the capital/labor ratio?

(b) If the saving fraction $s = .20$ and the initial capital/labor ratio is 1, describe the long-run behavior of the economy. What is consumption *per capita* in the steady state?

(c) What is meant by the golden rule of saving? For the steady state, calculate the real interest rate $r = f'(k)$, and compare this value to n. Relate the steady state to the golden rule.

Answer

(a)

$$f(k) = \frac{F(K,L)}{L} = \frac{\left(\frac{KL}{K+L}\right)}{L} = \frac{K}{K+L} = \frac{L}{L}\frac{\left(\frac{K}{L}\right)}{\left(\frac{K}{L}+1\right)} = \frac{k}{k+1}.$$

More directly,

$$f(k) = F(k,1) = \frac{k}{k+1}.$$

Solow's equation for the evolution of the capital/labor ratio is

$$\frac{\mathrm{d}k}{\mathrm{d}t} = s f(k) - nk.$$

The change in capital $\mathrm{d}k/\mathrm{d}t$ (capital deepening *per capita*) is the difference between $s f(k)$ (saving *per capita*) and nk (capital widening *per capita*).

(b) In the long run, the economy converges to steady-state growth. To solve for the steady-state capital/labor, set $\mathrm{d}k/\mathrm{d}t = 0$ and solve for k:

$$\begin{aligned} 0 &= s f(k) - nk \\ &= s\frac{k}{k+1} - nk. \end{aligned}$$

Cancelling k gives

$$0 = s\frac{1}{k+1} - n,$$

and solving gives

$$k = \frac{s}{n} - 1 = \frac{.20}{.04} - 1 = 4.$$

If initially $k = 1$, it will gradually increase until $k = 4$. In the long run, output *per capita* is $f(4) = \frac{4}{4+1} = .8$. Saving *per capita* is $sf(k) = .20 \times .8 = .16$, and consumption per capita is $.8 - .16 = .64$. Population, capital, and output all grow at rate $n = .04$.

(c) The golden rule of saving is to pick s to maximize steady state *per capita* consumption.

The following three conditions are equivalent:

a) golden-rule saving rate;

b) population growth = interest rate;

c) saving rate = capital income share.

The real interest rate is the marginal product of capital, the slope of the intensive production function:

$$\begin{aligned} r &= f'(k) \\ &= \frac{1}{k+1} - \frac{k}{(k+1)^2} \\ &= \frac{k+1}{(k+1)^2} - \frac{k}{(k+1)^2} \\ &= \frac{1}{(k+1)^2} \\ &= \frac{1}{(4+1)^2} \\ &= .04. \end{aligned}$$

Here we have the second condition, $n = r$, so $s = .20$ is the golden rule of saving.

Of course the third condition for the golden rule must also hold. Profit *per capita*

$$rk = .04 \times 4 = .16$$

equals saving *per capita*.

The capital income share is profit *per capita* divided by income *per capita*,

$$\frac{.16}{.80} = .20,$$

which is the saving rate.