Benjamin John Atchison

Research Statement

Let $\mathbb{F}_n$ denote the free group on the basis $\{x_0, \ldots, x_{n-1}\}$ and let α be an automorphism of $\mathbb{F}_n$. We call α a *shift automorphism* if

$$\alpha : x_0 \to x_1 \to \ldots \to x_{n-2} \to x_{n-1} \to w$$

for some $w \in \mathbb{F}_n$. We call w the *shift representing word* of the automorphism α. There is a clear analogy between shift automorphisms of free groups and companion matrices for polynomials, as well as a more subtle one between shift automorphisms and homeomorphisms of connected graphs. In the early 1990s, Addepalli worked with Turner to shed light on both of these analogies, and her thesis focuses on shift automorphisms having finite order. In 2005, a paper [AT05] was published that states the results of her research. Continuing along the same lines, my research deals with shift automorphisms of finite *outer* order k, i.e., $\alpha^k = \phi_u$, where $\phi_u \in Inn(\mathbb{F}_n)$ represents conjugation by some word u in $\mathbb{F}_n$.

If we view w in the free abelian group $\mathbb{F}_n^{ab}$, then the induced matrix M_α and resulting characteristic polynomial $\chi_\alpha(t)$ complete the correspondence between shift automorphims and companion matrices. For example, if $w = x_3 x_2 x_0^{-1} x_1 x_2^{-1} \in \mathbb{F}_4$, then

$$M_\alpha = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \chi_\alpha(t) = t^4 - t^3 - t + 1 = (t-1)^2(t^2+t+1). \tag{1}$$

In a paper for Roger Lyndon, [Cul84], Culler proved that any finite subgroup of $Out(\mathbb{F}_n)$ is realized by a group of homeomorphims of a compact graph. This result enables us to view every shift automorphism of finite outer order in a topological sense, rather than something purely algebraic. It also permits a reverse approach, in which we may analyze a particular family of compact connected graphs, along with an imposed homeomorphism, in the hopes that the induced automorphism on the fundamental group may be shift. Addepalli found some success in this approach, and eventually classified all shift automorphisms of finite order that are realized by a graph Γ and homeomorphism g of Γ, such that the quotient graph of Γ by g has geometric complexity one or two. Here, geometric (or graph) complexity refers to the number of edge orbits of Γ under g, i.e., the number of edges appearing in the quotient graph. Along these same lines, Turner and I have worked to classify all shift automorphisms of finite outer order that are realized by the pair (Γ, g), such that the quotient graph has geometric complexity one, and we have had some success in analyzing the cases where geometric complexity is greater than one. The fundamental distinction between my work and Addepalli's is the simple permission of the base point to move under g, resulting in finite *outer* order.

While focusing our attention on possible graph realizations of shift automorphisms, one cannot ignore the algebraic implications. The following observation, found in [AT05], has proven to be an invaluable aid.

If θ assigns the outer class to an automorphism and μ gives the induced matrix, then we have the following sequence.

$$\begin{array}{c} Aut(\mathbb{F}_n) \xrightarrow{\theta} Out(\mathbb{F}_n) \xrightarrow{\mu} GL_n(\mathbb{Z}) \xrightarrow{\chi} \mathbb{Z}[t] \\ \alpha \longmapsto \overline{\alpha} \longmapsto M_\alpha \longmapsto \chi_\alpha(t) \end{array}$$

The maps θ and μ are one-to-one on finite subgroups. If α has finite outer order k, then so does M_α, and by the Cayley-Hamilton Theorem, the minimal polynomial of M_α is a divisor of $t^k - 1$. In other words, if α is a shift automorphism of finite outer order k, then $\chi_\alpha(t) = \min_\alpha(t)$ is a product of distinct cyclotomic polynomials.

Consequently, if a pairing (Γ, g) is to realize a shift automorphism of finite outer order k, then, aside from the preliminary that g must have order k, the characteristic polynomial for the matrix induced from the action of g on $\pi_1(\Gamma, *)$ must also be a product of distinct cyclotomics. This means that the example in (1), although shift, cannot have finite outer order.

The cyclotomic requirement described above is necessary, but not sufficient in determining whether or not (Γ, g) realizes a shift automorphism of finite outer order. There is still the challenge of finding the right basis for $\pi_1(\Gamma, *)$ to obtain the single rational block associated with a shift automorphism. And even if this is possible, the exact makeup of the shift-representing word w is hardly obvious. For example, in (1) we could have $w = x_3 x_2 x_0^{-1} x_1 x_2^{-1}$ or $w = x_1 x_0^{-1} x_3$. Ultimately, our research touches upon the much larger question of whether or not two matrices are similar over the ring of integers, as illustrated in the following example, arising from a specific pairing (Γ, g), having a 2-edge quotient.

$$M_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\chi_1(t) = \chi_2(t) = t^7 - t^6 + t^5 + t^2 - t + 1$$
$$= (t+1)(t^2 - t + 1)(t^4 - t^3 + t^2 - t + 1) = c_2(t) c_6(t) c_{10}(t)$$

Research Objectives and Future Considerations

Since the onset of my research, it has always been an objective of mine to completely classify all pairs (Γ, g) with geometric complexity one, such that (Γ, g) realizes a shift automorphism α of finite outer order k. This includes explicit descriptions of the graph Γ, homeomorphism g, shift representing word w, minimal polynomial $\min_\alpha(t)$, and conjugator $u \in \mathbb{F}_n$, such that $\alpha^k = \phi_u$. With this virtually complete, there has been a natural progression to the geometric complexity two case and to further generalizations. One such generalization that Turner and I are working on is to show that given any compact connected graph Γ with homeomorphism g, there is a canonical choice of spanning tree T and basis for $\pi_1(\Gamma, *)$ that will determine whether or not the induced automorphism of g on $\pi_1(\Gamma, *)$ is indeed shift. Ultimately, the primary purpose of my research is to continue to develop and exploit the connection between shift automorphisms and the rational canonical form for matrices.

Still, a more intimate understanding of similarity of matrices over the integers is required for future research. This starts with a closer examination of a paper by Latimer and MacDuffee [LM33], which proves a one-to-one correspondence between ideal classes of square matrices over $\mathbb{Z}$ with minimal polynomial $f(t)$ and the ideal classes of $\mathbb{Z}[t]/f(t)$. Consequently, calculation of class numbers will also prove useful.

A majority of my research lends itself extremely well to the involvement of undergraduate students, as it utilizes key concepts and basic notions from Linear Algebra, Number Theory, Graph Theory, Topology, and the study of free groups. The numerous examples and calculations will spark interest and help to strengthen a student's education, and will undoubtedly lead to undergraduate research opportunities. I eagerly anticipate having the opportunity to introduce my students to the researcher's side of mathematics, which albeit essential, is seldom seen at the undergraduate level.

September 13, 2010

References

[AT05] V. Addepalli and E.C. Turner. Shift automorphims of finite order. In *Contemporary Mathematics: Geometric Methods in Group Theory*, pages 71–89. AMS, 2005.

[Cul84] M. Culler. Finite groups of outer automorphisms of a free group. In *Contemporary Mathematics: Contributions to Group Theory*, pages 197–208. AMS, 1984.

[LM33] C.G. Latimer and C.C. MacDuffee. A correspondence between classes of ideals and classes of matrices. In *The Annals of Mathematics, Second Series*, pages 313–316. Annals of Mathematics, 1933.