Introductory courses in symbolic logic focus on the description of formal models of reasoning, and on their application to the analysis and evaluation of arguments. However, we are more interested in studying certain (general/mathematical) properties of the formal systems themselves, not in applying these models to analyze and evaluate arguments. (Our topic is sometimes called metalogic, or the metatheory of logic.)

What is a formal system, or formal model of reasoning? From a contemporary perspective it is a language with a completely specified structure. In the first instance, structure means syntactic structure: a language is a collection of symbols together with rules for manipulating them; these rules not only specify which combinations of symbols are well-formed or grammatical expressions, they also determine which grammatical expressions are derivable from which. Of course, a language is more than this: it is also a medium for the expression of thoughts about an extra-linguistic reality, i.e. the symbols are given an interpretation and we are led to consider semantic properties as well. (Notions like grammaticality and derivability will seem arbitrary at best unless they are taken to reflect the semantic notions of meaningfulness and entailment.)

So, logicians are concerned with properties of language, albeit with a rather narrow range of them, and much of the contemporary interest of their subject derives from the success which logicians have had in understanding the relationship between syntax and semantics. Although there are branches of modern logic which do not fit this description, it supplies a convenient approach to the subject and helps to explain the reasons why logic has in recent years been taken up by mathematicians, computer scientists and linguists, as well as by philosophers.

For philosophers, language is the medium of thought, and analytic philosophy in the 20th century is distinguished by a conviction that the only approach to understanding the structure of thought is through a study of the language in which it is expressed. Furthermore, an understanding of the relationship between our thoughts and what they are about is central to some of the most fundamental problems of epistemology and metaphysics.

The distinctive feature of applications of logic to mathematics—whether to its foundations or to particular branches of the subject—is the introduction of considerations having to do with the language used to describe a mathematical structure. It is a surprising, fact that non-trivial properties of mathematical structures are revealed by such considerations.

The relevance of logic to computer science is related to the fact that machines can quite straightforwardly be programmed to recognize the syntactic features of expressions. Furthermore, since our use of language presupposes an ability to process syntax, i.e. to recognize when syntactic properties and relations hold and when they do not, it is not unreasonable to expect to be able to give an account of the procedures in question which is explicit enough to form the basis of a computer program able to accomplish the same task. An understanding of the relationship between syntax and semantics then opens up the possibility of a computers simulating the meaningful use of language. (This is only a part of the story. A number of programming languages have been based on the formal languages introduced by logicians. In addition, as a matter of historical fact, the theory of computation arose from the efforts of logicians to clarify the notion of computability in the context of the study of formal languages.)

The relationship between logic and linguistics is more problematic. Linguistics was, until relatively recently, much more a descriptive than an analytical science, and this has influenced the nature of linguistic theory. Logicians have been content to make radical simplifying assumptions about the languages they study for the sake of a smooth theory, whereas linguists view the richness and diversity of natural language as providing exactly the challenge that the theorist must meet. In recent times, however, there has been something of a rapprochement between the two fields, with logicians studying the logical properties of richer and more diverse languages and linguists borrowing some of the logician’s tools, especially semantical ones, in an effort to understand better how natural languages work.
The idea that all such questions can in principle be settled by calculation—in effect by manipulating syntactic forms—is only plausible if one accepts Leibniz’ conviction that all truths are analytic, and perhaps not even then.

What this argument assumes is that the ordinary laws of logic hold and that we can talk about the truth and falsity of English sentences within the English language itself. The first assumption can be weakened in various ways without dissolving the paradox. In particular, we need not assume bivalence, i.e. that every sentence is either true or false.

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what can only be shown. Nevertheless, they serve a heuristic purpose: he describes them as a ladder to be discarded once one has climbed to the top and achieved the correct logical point of view.

Picturesque though this metaphor of the ladder may be, it does not provide a satisfactory foundation for a serious discipline. In the 1920's Alfred Tarski established the beginnings of a scientific study of semantics. Amongst other results, he showed that in all except the simplest cases a distinction needs to be drawn between the language whose semantics was under discussion (the object language) and the language within which the semantic discussion is taking place (the meta language), and that the latter has to be more expressive in some sense than the former if paradox is to be avoided. As a result, logicians turned to the study of particular artificial languages and their logical properties, acknowledging that the results they obtained were relative to the language under discussion (being mentioned) and that this was separate from the language (being used) in which their discussion was taking place.

Although a number of outstanding technical problems were solved by this simple expedient, it represents something of a comedown for logic and creates problems of another kind. For example, logic seems to have lost its place as the fundamental discipline, except in rather an attenuated sense: we may still regard the logic of a language as more fundamental than any particular discourse—a scientific theory, say—formulated in that language, but that is a long way from being fundamental to thought itself—especially if the logic under discussion cannot be identified with the logic underlying our discussion of it. Similar problems arise in the justification of logical principles for, while it may be easy enough to justify the principles of inference employed in the language we are studying, our justification will itself presuppose the correctness of principles which are not themselves subject to scrutiny. (In light of these developments, it has sometimes been argued that logic no longer deserves to be counted as a branch of philosophy at all, but should rather be regarded as one more special science on a par with all the others by means of which we attempt to understand the world and ourselves.)

We won't pursue these very general issues. Rather, we propose to concentrate upon the logical properties of a particular language or, more precisely, of a family of languages—namely those of (classical) first-order predicate logic—and on formal theories based upon them. These are the languages one usually encounters in an introductory logic course. Their non-logical constants may include names for individuals, symbols expressing properties of or relations between these individuals, and perhaps symbols for functions from individuals to individuals. Their logical constants will consist of the usual truth-functional connectives (negation, conjunction, disjunction and implication) together with universal and existential quantifiers over individuals, and perhaps the relation of identity. (In fact, for pedagogic reasons, we'll begin by discussing a more restricted language, that of propositional logic, built up from truth-functional connectives and sentence letters.)

These languages have their origin in Frege's Begriffsschrift in the sense that they can be viewed as fragments of the language considered therein. Another problem I wish to avoid for the most part is the justification for singling out this family of languages for attention. Historically, they were extrapolated from the more expressive languages devised by Frege and Russell because they were thought to be adequate for the expression of mathematical reasoning and because they possess certain useful properties which stronger languages turn out to lack. (We shall have more to say about some of these properties below.) Furthermore, as those who have encountered them in the context of an elementary logic course can attest, they are sufficiently expressive to do justice to a wide variety of arguments that we can
formulate in English. Whether their syntactic structure reflects the structure of thought, or even the logical structure of a natural language like English, is a question we won’t address. What makes them worth studying, especially in an elementary course on logic, is that on the one hand they are sufficiently expressive to be useful and interesting and on the other that most of the important techniques developed by logicians receive their simplest employment in the analysis of the logical properties of these first order languages. In short, they are complicated enough to be interesting and simple enough to be manageable. The remainder of the course, except for an occasional digression, will be devoted exclusively to their study.

On the face of it, this is a different approach to the subject to the one taken by Zalabardo in his introduction. He treats logic as dealing with propositions, where these are identified as “ways for things to be” and described as objects of beliefs, desires and other so-called propositional attitudes. Logic is the study of how structural properties of propositions, the way in which they are built up from their components, generate relationships between their truth values. (Presumably, a “way for things to be” is true if it is the way things are.) An important example of such a relationship, perhaps the central one in logic, is logical consequence: one proposition is a logical consequence of some others if it has to be true whenever all of them are true.

He then focuses on a particular class of propositions, ones which “can be characterized in terms of which individuals instantiate certain properties and relations”. These he calls first-order propositions, so that first-order logic is the study of how “the pattern according to which properties, relations and individuals figure in a collection of first-order propositions sometimes generates links between their truth values.”

Subsequently, he goes on to introduce a formal language as an aid in studying properties of these propositions and, henceforth, the formulas of this language become the principal objects of study. What distinguishes the language in question, according to Zalabardo, is that its sentences are built from their components “according to principles which neatly correlate” with the ways in which first-order propositions are composed of theirs.

In other words, Zalabardo treats the sentences of a formal language as surrogates for propositions, where the latter are given independently of the sentences which express them. However, since we have no access to propositions except through the medium of language, I think it is preferable to begin simply by choosing a language and studying its syntactic and semantic properties. Only after we have done so is it appropriate to ask whether we need to introduce such abstractions as propositions to give significance to our results and, if we do, whether these can plausibly be claimed to have a structure similar to the sentences of our language. Although these differences of approach are not insignificant, they do not really affect the development of the theory. Whatever the starting point, both approaches quickly lead to the same place—namely logic as the study of certain properties of formal languages.

Some philosophers have argued that they do encompass the logic of mathematical proof and some—like Donald Davidson—have even argued that they are our best guide to the logical structure of natural language.
The first step is to describe the syntax (or grammar) of such a language (actually, of a propositional language, first)—in particular, to define what it means to be a grammatical sentence of the language—and establish some simple syntactic properties. We will then go on to discuss semantics. The basic semantic property is that of truth, more precisely, of truth under an interpretation. To define what it means for a sentence to be true under an interpretation, we will first have to explain the notion of an interpretation. Because this is explained in set-theoretic terms, we will have to begin with a preliminary discussion of some elementary properties of sets. (The language of set theory has other uses too: it’s needed to describe syntax, for example, and to express some of the deeper properties of formal systems.)

The definition of truth for a language will, like many other definitions we shall give, be an inductive one. This is important because, to every inductively defined property, there corresponds a principle of proof by induction on the definition that enables us to prove assertions about all things which have the property. The use of such principles is so ubiquitous in logic that it will be worth our while to consider them separately as part of our preliminary discussion.4

Once we have defined truth under an interpretation, we can go on to define other important semantic notions in terms of it. (For example, logical validity can be defined as truth under every interpretation.) We will then be in a position to discuss what is central to our course, namely the relationship between the syntax and semantics of these first order languages. An important goal is to attempt to characterize certain fundamental semantic properties in syntactic terms. This is of interest not just because it may establish the computability (in some suitable sense) of some of these semantic properties, but also because we understand syntax so much better than semantics: language as a system of symbols is scarcely mysterious at all5; how language can be about extra-linguistic matters is still something of a problem. So, if we are able to effect some sort of reduction of a semantic notion to a syntactic one, while that might not clear up the problem, it would at least enable us to avoid it (and perhaps clarify our understanding of the semantic property in question). Notice incidentally that were we able to give a syntactic characterization of truth (under some interpretation) we would have succeeded in realizing Leibniz dream of being able to resolve all disputes—in this limited context, at least—by calculation.

We shall begin by considering not truth, but validity, and will prove that the set of (classically) valid sentences of a first order language can indeed be characterized in syntactic terms. This result is the so called Completeness Theorem for first order logic. It is the most fundamental result about elementary

4 To get a rough idea of what is involved here, consider what is perhaps the simplest inductively defined property, namely that of being a natural number. The definition proceeds by stipulating that 0 is such a number, that (for any n) if n is a natural number, so is its successor, and that these are the only natural numbers. Corresponding to this definition is a principle of proof which is known as (weak) mathematical induction that enables us to prove assertions about all natural numbers. It asserts that if we can establish that 0 has a certain property and that, whenever a natural number n has the property, its successor must have the property as well, then we can conclude that all natural numbers have the property in question. Notice how the principle of proof mirrors the original definition.

5 This is not to deny that there is much we don’t know about the syntax of natural language and how it is processed by speakers.
logic and gives the course half of its name. We will then go on to consider particular examples of first order languages and their intended interpretations. The question that arises in each case is whether the notion of truth under the intended interpretation can be given a syntactic characterization similar in kind to that of validity. The most interesting of these languages is perhaps the language of arithmetic, whose intended interpretation is the structure consisting of the natural numbers: 0, 1, 2, 3, ... together with the familiar operations of addition and multiplication. Here it turns out that the notion of truth cannot be captured by any appropriate syntactic property of sentences. From this will follow the undecidability of any theory of arithmetic formulated in the language, in the sense that the set of theorems of such a theory will fail to coincide with the set of sentences true of the natural numbers. In the best possible case there will be some true sentences of arithmetic which are neither provable nor refutable within the theory. This is the import of Gödel's famous (first) Incompleteness Theorem.