Review Problems for Exam 1, AMAT 327

1. For each of the following relations determine whether they are reflexive, symmetric, transitive or equivalence.

(a) For $a, b \in \mathbb{Z}$, define $a \sim b$ if $a + b$ is even.

(b) For $a, b \in \mathbb{Z}$, define $a \sim b$ if $a + b$ is odd.

(c) On $\mathbb{R}^*$ (nonzero real numbers), define $a \sim b$ if $\frac{a}{b} \in \mathbb{Q}$.

(d) On $\mathbb{R}^*$, define $a \sim b$ if $\frac{a}{b} \in \mathbb{Z}$.

2. On the set of all ordered pairs of positive real numbers define $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 y_2 = x_2 y_1$. Show that this defines an equivalence relation. Give a geometric interpretation of congruence classes of $\sim$.

3. Compute $(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})^n$, where $n = 3$, 50 and 100.

4. Construct a subgroup H with six elements inside $\mathbb{Z}_{18}$. Determine all H-cosets in $\mathbb{Z}_{18}$.

5. Show that there is no element of order 4 in S_3. Show there is no element of order 5 in S_4.

6. In S_4, find a cyclic subgroup of order 4. Is there a non-cyclic subgroup of order 4? If yes, give an example.

7. Show that every subgroup of a cyclic group is cyclic.

8. Let G be a group of order 4. Show that G must be abelian.

9. Give an example of group G and two subgroups H_1 and H_2, such that $H_1 \cup H_2$ is not a subgroup of G.

10. Show that $Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}$ is a subgroup of G.

11. Find the center of the dihedral group with 10 elements. Generalize this for all dihedral groups with $2n$ elements where n is odd.

12. For vectors (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in \mathbb{R}^3$, the cross product is defined by

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2).$$

Is $\mathbb{R}^3$ a group under this multiplication?

13. Let $G = \mathbb{Q}^*$ (nonzero rational numbers). Define a new multiplication $a * b = \frac{ab}{2}$. Does $(G, *)$ have a group structure?

14. Interpret $f(x) = \log(x)$ as an isomorphism among two abelian groups.

15. Which of the following functions are group homomorphisms? For each homomorphism determine whether it is (a) epi, (b) mono or (c) iso.

(i) $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$, $f(m) = 5 + m$.

(ii) $f : (\mathbb{Q}^*, \cdot) \to (\mathbb{Q}^*, \cdot)$, $f(x) = \frac{1}{x}$.

(iii) $f : S_3 \to S_3,\ f(\sigma) = \sigma^{-1}$.

(iv) $f : \mathbb{Z} \to \mathbb{Z}_n,\ f(a) = [a]$.

16. Suppose that H is a proper subgroup of $\mathbb{Z}$ that contains 18, 30 and 40. Find m such that $H = m\mathbb{Z}$.

17. Let $\varphi(m)$ be the Euler's function. Prove Euler's theorem $a^{\varphi(m)} \equiv 1 (\text{mod } m)$ for every $(a, m) = 1$.

In Problems 18-21, GL_2 denotes the group of invertible 2×2 matrices with real entries under the usual multiplication of matrices.

18. Show that the upper triangular 2×2 matrices (those with 0 in the lower-left corner) with non-zero entries on the diagonal is a subgroup of GL_2

Solution: We multiply two such matrices:

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix}$$

But now, $aa' \neq 0$ and $cc' \neq 0$, so the set is closed under multiplication.

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{ca} \\ 0 & \frac{1}{c} \end{bmatrix}.$$

Thus the inverses are also inside.

19. Let $H = \{ \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}, \theta \in \mathbb{R}\}$. Show that H is a subgroup of GL_2.

Solution: Observe that

$$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} cos(\theta') & -sin(\theta') \\ sin(\theta') & cos(\theta') \end{bmatrix}$$

$$= \begin{bmatrix} cos(\theta)cos(\theta') - sin(\theta)sin(\theta') & -sin(\theta)cos(\theta') - cos(\theta)sin(\theta') \\ sin(\theta)cos(\theta') + cos(\theta)sin(\theta') & cos(\theta)cos(\theta') - sin(\theta)sin(\theta') \end{bmatrix} = \begin{bmatrix} cos(\theta + \theta') & -sin(\theta + \theta') \\ sin(\theta + \theta') & cos(\theta + \theta') \end{bmatrix}$$

so this set is closed under multiplication. The inverse of $\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$ is $\begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(-\theta) & cos(-\theta) \end{bmatrix}$ which is also H. Therefore H is a subgroup.

20. Show that 2×2 with positive determinant form a subgroup of GL_2.

Solution: This follows from the formula $det(AB) = det(A)det(B)$. Clearly, product of two matrices with positive determinant results with a matrix with positive determinant. Because of $det(A^{-1}) = \frac{1}{det(A)}$ the inverse of a A has positive determinant. So we have a subgroup.

21. Show that $G = \{\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}, m \neq 0\}$ forms a group. Identify the center of G (center of G is the set of elements commuting with all elements in the group).

Solution:

$$\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} mm' & b + mb' \\ 0 & 1 \end{bmatrix},$$

so G is closed under multiplication (observe $mm' \neq 0$). Multiplication of matrices is associative. The unit matrix belong to G (take $m = 1$ and $b = 0$). Finally,

$$\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{m} & -\frac{b}{m} \\ 0 & 1 \end{bmatrix},$$

which is again in G.

22. Let G be an abelian group of 15 elements in which you can find an element of order 3 and an element of order 5, then G must be cyclic.

23. In the multiplicative group $\mathbb{C}^*$ (nonzero complex numbers), find the order of elements $-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

24. Prove that in a finite group G, $a^{|G|} = e$ for every $a \in G$.

25. Let G be closed under a multiplication $*$ and associative. Show that $a * x = a$ and $y * a = a$ for every $a \in G$ implies $x = y$ (in other words left and right identity must be equal).

26. Give three examples A, B and C of groups of order 8 such that (i) A is abelian and non-cyclic, (ii) B is abelian and cyclic and (c) C is nonabelian. (Hint: for part (a) consider U_n).

27. Let A, B be subgroups of an abelian group G. Show that $AB = \{ab : a \in A, b \in B\}$ is a subgroup of G. Show that this is not true for nonabelian groups (Hint: consider a dihedral group).

28. Show that $\mathbb{Z}$ (under the usual addition of integers) admits an infinite sequence of distinct subgroups H_i, $i \geq 1$, such that

$$\cdots \subset H_2 \subset H_1 \subset \mathbb{Z}.$$

29. Denote by G the set of all functions $T_{a,b}(r) = ar + b$, $a \neq 0$ (under the usual composition). Determine the left H-cosets in G, where H is the subgroup consisting of $T_{1,a}$ functions

30. In the dihedral group of 20 elements $G = \{f^i h^j : i = 0, 1, j = 0, ..., 9\}$ simplify the expression

$$fh^5fh^3fhhfh^{-3}h^{-8}f.$$

31. Identify the *subgroups* of $(\mathbb{Z}_{12}, +)$.

32. Compute the order of each element in U_{10}. Is U_{10} cyclic?

33. Describe all subgroup of $\mathbb{Z}$ (under the usual addition). Find all distinct subgroups $H \subset \mathbb{Z}$ such that $100\mathbb{Z} \subset H \subset 5\mathbb{Z}$.

34. Let G be a group in which $ga = ag^{-1}$ and $g^n = 1$ for some $a, g \in G$. Show that $g^i a = ag^{n-i}$ for any $i \in \mathbb{N}$.

35. In the group U_{36} consider $H = \{[x] : x \equiv 1(mod\ 4)\}$. Show that H is a subgroup U_{36}. Determine the left (or right) H-cosets of U_{36}

36. Show that if $H_1 \neq H_2$ are subgroups of a group G and $|H_1| = |H_2| = 5$, then $|H| \cap |H_2| = \{e\}$.

37. Let $\mathbb{R}\backslash\{1\}$ be the set of real numbers without 1 with binary operators $*$ such that

$$x * y = x + y - xy.$$

Show that this set has a group structure.

38. Show that any infinite group has infinitely many subgroups.

39. Show that U_5 and $\mathbb{Z}_4$ are isomorphic.