A Free Resolution from Simplicial Complexes

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Let $R = \mathbb{k}[x_1, \ldots, x_m]$ be a polynomial ring in m variables over a field $\mathbb{k}$ with the standard grading and L a finite multigraded ($\mathbb{Z}^m$ graded) R-module.

PROBLEM: Is there a combinatorial object associated to L that completely describes a free resolution of L?

For a monomial ideal I, Taylor constructed a free resolution of $L = R/I$ from a simplicial complex associated to I.

Simplicial Complex

A simplicial complex Δ on a finite set V is a collection of subsets of V so that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$

We call the elements of V, vertices. We call the subsets of Δ faces. A face not properly contained in another face is called a facet.
If $F \in \Delta$ is a face then $\dim F = |F| - 1$.
The $\dim \Delta = \max\{\dim F : F \in \Delta\}$.

A simplicial complex is pure if all facets are equidimensional.

A construction by Tchernev gives a free resolution (not necessarily minimal) of a multigraded module L of the form

$$0 \to T_\lambda(\Phi, S) \xrightarrow{\Phi_\lambda} T_{\lambda-1}(\Phi, S) \to \cdots \to T_1(\Phi, S) \xrightarrow{\Phi_1} T_0(\Phi, S) \to 0$$

where

$$E \xrightarrow{\Phi} G \longrightarrow L \longrightarrow 0$$

is a free minimal multigraded presentation of L, and S is a multihomogeneous basis of E.

Tchernev defines a matrix ϕ from a minimal free presentation of L

$$E \xrightarrow{\Phi} G \longrightarrow L \longrightarrow 0$$

by tensoring with $\mathbb{k} = R/(x_1 - 1, \ldots, x_n - 1)$. This vector space map $\Phi \otimes 1 : E \otimes \mathbb{k} \longrightarrow G \otimes \mathbb{k} = W$ restricts to a set map $\phi : S \longrightarrow W$ inducing a vector space map $\phi : U_S \longrightarrow W$.

$$\begin{array}{ccc} E \otimes \mathbb{k} & \xrightarrow{\Phi \otimes 1} & G \otimes \mathbb{k} \\ \downarrow\cong & & \downarrow= \\ U_S & \xrightarrow{\phi} & W \end{array}$$

$$
\begin{array}{ccccccccc}
0 \to T_\lambda(\Phi,S) & \xrightarrow{\Phi_\lambda} & T_{\lambda-1}(\Phi,S) & \to \cdots \to & T_1(\Phi,S) & \xrightarrow{\Phi_1} & T_0(\Phi,S) \to 0 \\
\downarrow \otimes \mathbb{k} & & \downarrow \otimes \mathbb{k} & & \downarrow \otimes \mathbb{k} & & \downarrow \otimes \mathbb{k} \\
0 \longrightarrow T_S(\phi) \longrightarrow & & \bigoplus_{l_A=\lambda-3} T_A(\phi) \to & \cdots & \to U_S & \xrightarrow{\phi} & W \longrightarrow 0
\end{array}
$$

Matroids

A matroid M on a finite set S is a pure simplicial complex on the vertex set S, if each subcomplex $\Delta_A = \{F \in \Delta : F \subseteq A\}$ is pure for all $A \subseteq S$.

We call the faces the independent sets of the matroid.

The facets are the bases of the matroid.

The minimal non-faces are called the circuits of the matroid.

The union of circuits are called the t-flats of the matroid.

The rank of a matroid is the number $r_M = \dim \Delta + 1 = |F|$, where F is a facet.

The level of a matroid is the number $l_M = |S| - r_M - 1$.

The dual matroid M^* to M has vertex set S and facets are $S - F$ where F is a facet of M.

Matroid Operations

The restriction of a matroid M to A, $M|A$ is defined to be the subcomplex

$$\Delta_A = \{F \in M : F \subseteq A\}.$$

Notation: $M|(S-a) = M \backslash a$

The contraction of a matroid M on A, $M.A$ is the subcomplex

$$\Delta_{M.A} = \{F \subseteq (S-A) : F \cup B \in M, \text{ for some } B \subseteq A\}.$$

Notation: $M.(S-a) = M/a$

This matrix ϕ gives a matroid M on $|S|$ vertices where the facets are precisely those A satisfying $\dim(\phi(A)) = |A|$.

A matroid M defined in this way is called <u>representable</u>, and ϕ is called a <u>representation</u> of M.

Recall from the construction that we have a matrix $\phi : U_S \longrightarrow W$.

$$0 \longrightarrow T_S(\phi) \longrightarrow \bigoplus_{l_A=\lambda-3} T_A(\phi) \rightarrow \quad \cdots \quad \rightarrow U_S \xrightarrow{\phi} W \longrightarrow 0$$

Theorem(B.)
Let M be a representable matroid over a vector space $\mathbb{k}$ with representation ϕ. Then

$$\dim_{\mathbb{k}}(T_S(\phi)) = \beta(M),$$

where $\beta(M)$ is Crapo's beta invariant.

Crapo's Beta Invariant

$$\beta(M) := (-1)^{r_M} \sum_{A \subseteq S} (-1)^{|A|} r_A$$

Basic facts:

1. The number $\beta(M) \geq 0$ for every matroid M.

2. $\beta(M) = 0$ precisely when M is disconnected.

3. For any matroid M with dual M^*, $\beta(M) = \beta(M^*)$.

4. $\beta(M/a) + \beta(M \backslash a) = \beta(M)$.

For a matroid M, with $|S| = n$, we can define a linear ordering on the vertices by making $S = \{1, 2, \ldots, n\}$ and $1 < 2 < \ldots < n$. We call a matroid with this linear ordering an ordered matroid.

Broken Circuit Complex

Given an ordered matroid M on S.

A broken circuit is obtained from a circuit by deleting its smallest element.

The family of all subsets of S that contain NO broken circuits is called the broken circuit complex of M, written $BC(M)$.

The family of all subsets of $S - 1$ that contain NO broken circuits is called the reduced broken circuit complex of M, written $\overline{BC}(M)$.

Theorem(Brylawski) Let M be an ordered matroid of rank r. Then

1. $\overline{BC}(M) \subseteq BC(M) \subseteq M$.

2. $BC(M)$ is a pure $(r-1)$ dimensional complex of M.

3. $\overline{BC}(M)$ is a pure $(r-2)$ dimensional complex of M.

4. $BC(M)$ is a cone over $\overline{BC}(M)$ with apex 1 (smallest element of S), whose facets we call nbc-basis.

Theorem (Björner) For an ordered matroid M on a set S, the simplicial complexes $M, BC(M)$, and $\overline{BC}(M)$ have a canonical set of basic cycles for the reduced homology group

$$\widetilde{H}_d(\Delta;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^q & \text{if } d = \dim \Delta \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Ziegler) Let M be an ordered matroid of rank r. Then $\overline{BC}(M)$ has top-dimensional reduced homology

$$\widetilde{H}_{r-2}(\overline{BC}(M); \mathbb{Z}) \cong \mathbb{Z}^{\beta(M)}.$$

Moreover, the canonical set of basic cycles for it is given by the set

$$\{B \backslash 1 : B \in \beta nbc(M)\}.$$

The β-system of M is the collection of bases

$$\beta nbc(M) = \{B \in \text{nbc-basis} : IA(B) = \{1\}\}.$$

An element $p \in B$ is internally active $[p \in IA(B)]$ if p is the smallest element in the circuit of M^* that is contained in $(S - B) \cup \{p\}$.

Theorem (B.)
Let M be a matroid on a set S with dual M^*. Then there is a canonical isomorphism

$$\widetilde{H}_{r-2}(\overline{BC}(M^*); \Bbbk) \xrightarrow{\cong} T_S(\phi).$$

$$
\begin{array}{ccccccccccccc}
 & & \widetilde{H}_{r-2}(\overline{M^*}) & & \bigoplus_{l_A=\lambda-3} \widetilde{H}_{r-2}(\overline{M^*/A}) & & & & \mathbb{k}^{|S|} & & W & & \\
 & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & T_S(\phi) & \longrightarrow & \bigoplus_{l_A=\lambda-3} T_A(\phi) & \longrightarrow & \cdots & \longrightarrow & U_S & \longrightarrow & W & \longrightarrow & 0
\end{array}
$$

Theorem(B.)

Let M be a matroid on a set S with dual M^*. If $m = max(S)$ is not a loop then the following diagram commutes,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{H}_{r-2}(\overline{M^*\backslash m}) & \longrightarrow & \widetilde{H}_{r-2}(\overline{M^*}) & \longrightarrow & \widetilde{H}_{r-3}(\overline{M^*/m}) & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow & & \\
0 & \longrightarrow & T_{S/m}(\phi) & \longrightarrow & T_S(\phi) & \longrightarrow & T_{S\backslash m}(\phi) & \longrightarrow & 0
\end{array}$$