Longitudinal modeling of social networks

Social networks: structures of relations between social actors.
Examples:

- friendship between school children
- friendship between colleagues
- advice between colleagues
- alliances between firms
- alliances and conflicts between countries
- etc.......

These can be represented mathematically by graphs or more complicated structures.
**Why are ties formed?**

There are many recent approaches to this question leading to a large variety of mathematical models for network dynamics.

The approach taken here is for statistical inference:

- a flexible class of stochastic models that can adapt itself well to a variety of network data and can give rise to the usual statistical procedures: estimating, testing, model fit checking.

Examples of research questions in social network analysis:

- Is there a tendency in friendship toward transitivity? (‘friends of my friends are my friends’)
- Does ethnic background have an effect on friendship, controlling for reciprocity and transitivity?
- What is the role of friendship between adolescents in smoking initiation?
- Is advice giving / receiving related to status?
- Is there a hierarchy in advice?
- Do strategic alliances follow earlier contacts between board members?
Data collection designs

Many designs possible for collecting network data; e.g.,

1. Non-longitudinal: all ties on one predetermined node set;
2. Longitudinal: panel data with \( M \geq 2 \) data collection points, at each point all ties on the predetermined node set;
3. Longitudinal: continuous observation of all ties on one node set;
4. Incomplete continuous longitudinal (inter-firm ties): as above, but without recording termination of ties;
5. Snowballing: node set not predetermined (e.g., small world experiment).

Statistical procedures will depend on data collection design.

In some of such questions, networks are \textit{independent variables}. This has been the case in many studies for explaining well-being (etc.); this later led to studies of network resources, social capital, solidarity, in which the network is also a \textit{dependent variable}.

Networks are dependent as well as independent variables: intermediate structures in macro–micro–macro phenomena.
Here: focus first on networks as dependent variables, then on mutual dependence networks and behavior (‘behavior’ stands here also for other individual attributes).

Single observations of networks are snapshots, the results of untraceable history. *Everything depends on everything else.*

Therefore, explaining them has limited importance. Longitudinal modeling offers more promise for understanding. *The future depends on the past.*

1. Networks as dependent variables

Repeated measurements on social networks: at least 2 measurements (preferably more).

*Data requirements:*

The repeated measurements must be close enough together, but the total change between first and last observation must be large enough in order to give information about rules of network dynamics.
Example: Studies Gerhard van de Bunt

Longitudinal study: panel design.

► Study of 32 freshman university students, 7 waves in 1 year.

This data set can be pictured by the following graphs (arrow stands for ‘best friends’).

Friendship network time 1.

Average degree 0.0; missing fraction 0.0.
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Friendship network time 2.

Average degree 0.7; missing fraction 0.06.

Friendship network time 3.

Average degree 1.7; missing fraction 0.09.
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Friendship network time 4.

Average degree 2.1; missing fraction 0.16.

Friendship network time 5.

Average degree 2.5; missing fraction 0.19.
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Friendship network time 6.
Average degree 2.9; missing fraction 0.04.

Friendship network time 7.
Average degree 2.3; missing fraction 0.22.
Which conclusions can be drawn from such a data set?

Dynamics of social networks are complicated because “network effects” are endogenous feedback effects: e.g., reciprocity, transitivity, popularity, subgroup formation.

For statistical inference, we need models for network dynamics that are flexible enough to represent the complicated dependencies in such processes; while satisfying also the usual statistical requirement of parsimonious modelling: complicated enough to be realistic, not more complicated than empirically necessary and justifiable.

For a correct interpretation of empirical observations about network dynamics collected in a panel design, it is crucial to consider a model with latent change going on between the observation moments.

E.g., groups may be regarded as the result of the coalescence of relational dyads helped by a process of transitivity (“friends of my friends are my friends”).

Which groups form may be contingent on unimportant details; that groups will form is a sociological regularity.

Therefore:
use dynamic models with continuous time parameter: time runs on between observation moments.
Longitudinal modeling of social networks

An advantage of using continuous-time models, even if observations are made at a few discrete time points, is that a more natural and simple representation may be found, especially in view of the endogenous dynamics. (cf. Coleman, 1964).

No problem with irregularly spaced data.

This has been done in a variety of models:
For discrete data: cf. Kalbfleisch & Lawless, JASA, 1985;
for continuous data:
mixed state space modelling well-known in engineering,
in economics e.g. Bergstrom (1976, 1988),
in social science Tuma & Hannan (1984), Singer (1990s).

Purpose of statistical inference:
investigate network evolution (dependent var.) as function of

1. structural effects (reciprocity, transitivity, etc.)
2. explanatory actor variables (independent vars.)
3. explanatory dyadic variables (independent vars.)

simultaneously.
By controlling adequately for structural effects, it is possible to test hypothesized effects of variables on network dynamics (without such control these tests would be incomplete).

The structural effects imply that the presence of ties is highly dependent on the presence of other ties.
Principles for this approach to analysis of network dynamics:

1. use simulation models as *models for data*
2. comprise a random influence in the simulation model to account for ‘unexplained variability’
3. use methods of statistical inference for probability models implemented as simulation models
4. for panel data: employ a continuous-time model to represent unobserved endogenous network evolution
5. condition on the first observation and do not model it: no stationarity assumption.

Notation and assumptions

1. *Actors* \( i = 1, \ldots, n \) (individuals in the network), pattern \( X \) of *ties* between them: one binary network \( X \); \( X_{ij} = 0, \) or 1 if there is no tie, or a tie, from \( i \) to \( j \). Matrix \( X \) is *adjacency matrix* of digraph. \( X_{ij} \) is a *tie indicator* or *tie variable*.
2. Exogenously determined independent variables:
   - actor-dependent covariates \( v \), dyadic covariates \( w \).
   These can be constant or changing over time.
3. Continuous time parameter \( t \), observation moments \( t_1, \ldots, t_M \).
4. Current state of network \( X(t) \) is dynamic constraint for its own change process: Markov process.
Actor-based model:

5. The actors control their outgoing ties.

6. The ties have inertia: they are states rather than events.
   At any single moment in time,
   only one variable $X_{ij}(t)$ may change.

7. Changes are modeled as
   choices by actors in their outgoing ties,
   with probabilities depending on 'objective function'
   of the network state that would obtain after this change.

The change probabilities can (but need not)
be interpreted as arising from goal-directed behavior,
in the weak sense of myopic stochastic optimization.

Assessment of the situation is represented by
objective function, interpreted as
'that which the actors seem to strive after in the short run'.

Next to actor-driven models,
also tie-driven models are possible.
At any given moment, with a given current network structure, the actors act independently, without coordination. They also act one-at-a-time.

The subsequent changes (‘micro-steps’) generate an endogenous dynamic context which implies a dependence between the actors over time; e.g., through reciprocation or transitive closure one tie may lead to another one.

This implies strong dependence between what the actors do, but it is completely generated by the time order: the actors are dependent because they constitute each other’s changing environment.

The change process is decomposed into two sub-models, formulated on the basis of the idea that the actors \( i \) control their outgoing ties (\( X_{i1}, \ldots, X_{im} \)):

1. waiting times until the next opportunity for a change made by actor \( i \):
   rate functions;

2. probabilities of changing (toggling) \( X_{ij} \), conditional on such an opportunity for change:
   objective functions.

The distinction between rate function and objective function separates the model for how many changes are made from the model for which changes are made.
This decomposition between the timing model and the model for change can be pictured as follows:

At randomly determined moments \( t \), actors \( i \) have opportunity to change a tie variable \( X_{ij} \):

\[ \text{micro step.} \]

(Actors are also permitted to leave things unchanged.)

Frequency of micro steps is determined by \textit{rate functions}.

When a micro step is taken,
the probability distribution of the result of this step depends on the \textit{objective function}:
higher probabilities of moving toward new states that have higher values of the objective function.

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**Specification: rate function**

\textit{‘how fast is change / opportunity for change ?’}

Rate of change of the network by actor \( i \) is denoted \( \lambda_i \):
expected frequency of changes by actor \( i \) between observations.

Simple specification: rate functions are constant within periods.

More generally, rate functions can depend on observation period \((t_{m-1}, t_m)\),
actor covariates, and network position (degrees etc.),
through an exponential link function.

Formally, for a certain short time interval \((t, t + \epsilon)\),
the probability that this actor randomly gets an opportunity to change one of his/her outgoing ties, is given by \( \epsilon \lambda_i \).
Specification: objective function

‘what is the direction of change?’

The objective function \( f_i(\beta, x) \) indicates preferred ‘directions’ of change. 
\( \beta \) is a statistical parameter, \( i \) is the actor (node), \( x \) the network.

When actor \( i \) gets an opportunity for change, 
he has the possibility to change one outgoing tie variable \( X_{ij} \), 
or leave everything unchanged.

By \( x(i \sim j) \) is denoted the network obtained 
when \( x_{ij} \) is changed (‘toggled’) into \( 1 - x_{ij} \).
Formally, \( x(i \sim i) \) is defined to be equal to \( x \).

Conditional on actor \( i \) being allowed to make a change, 
the probability that \( X_{ij} \) changes into \( 1 - X_{ij} \) is

\[
p_{ij}(\beta, x) = \frac{\exp \left( f_i(\beta, x(i \sim j)) \right)}{\sum_{h=1}^{n} \exp \left( f_i(\beta, x(i \sim h)) \right)},
\]

and \( p_{ii} \) is the probability of not changing anything.

Higher values of the objective function indicate 
the preferred direction of changes.
One way of obtaining this model specification is to suppose that actors make changes such as to optimize the objective function $f_i(\beta, x)$ plus a random disturbance that has a Gumbel distribution, like in random utility models in econometrics: myopic stochastic optimization, multinomial logit models.

Actor $i$ chooses the “best” $j$ by maximizing

$$f_i(\beta, x(i \sim j)) + U_i(t, x, j).$$

(↑ random component)

(with the formal definition $x(i \sim i) = x$).

For a convenient distributional assumption, $(U$ has type 1 extreme value = Gumbel distribution) given that $i$ is allowed to make a change, the probability that $i$ changes the tie variable to $j$, or leaves the tie variables unchanged (denoted by $j = i$), is

$$p_{ij}(\beta, x) = \frac{\exp (f(i, j))}{\sum_{h=1}^{n} \exp (f(i, h))}$$

where

$$f(i, j) = f_i(\beta, x(i \sim j))$$

and $p_{ii}$ is the probability of not changing anything.

This is the multinomial logit form of a random utility model.
Objective functions will be defined as sum of:

1. *evaluation function* expressing satisfaction with network;
2. *endowment function* 
   expressing aspects of satisfaction with network that are obtained ‘free’ but are lost at a value (to allow asymmetry between creation and deletion of ties).

Evaluation function and endowment function modeled as linear combinations of theoretically argued components of preferred directions of change. The weights in the linear combination are the statistical parameters.

The focus of modeling is first on the evaluation function; then on the rate and endowment functions.

The objective function does not reflect the eventual ‘utility’ of the situation to the actor, but short-time goals following from preferences, constraints, opportunities.

The evaluation and endowment functions express how the dynamics of the network process depends on its current state.
Intensity matrix

This specification implies that $X$ follows a continuous-time Markov chain with intensity matrix

$$q_{ij}(x) = \lim_{dt \downarrow 0} \frac{P\{X(t + dt) = x(i \rightarrow j) \mid X(t) = x\}}{dt} \quad (i \neq j)$$

given by

$$q_{ij}(x) = \lambda_i(\alpha, \rho, x) p_{ij}(\beta, x).$$

Computer simulation algorithm

for arbitrary rate function $\lambda_i(\alpha, \rho, x)$

1. Set $t = 0$ and $x = X(0)$.
2. Generate $S$ according to the exponential distribution with mean $1/\lambda_+(\alpha, \rho, x)$ where

$$\lambda_+(\alpha, \rho, x) = \sum_i \lambda_i(\alpha, \rho, x).$$

3. Select $i \in \{1, \ldots, n\}$ using probabilities

$$\frac{\lambda_i(\alpha, \rho, x)}{\lambda_+(\alpha, \rho, x)}.$$
4. Select \( j \in \{1, \ldots, n\}, j \neq i \) using probabilities \( p_{ij}(\beta, x) \).

5. Set \( t = t + S \) and \( x = x(i \sim j) \).

6. Go to step 2 
   (unless stopping criterion is satisfied).

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**Model specification:**

Simple specification: only evaluation function; no endowment function, periodwise constant rate function.

Evaluation function \( f_i \) reflects network effects (endogenous) and covariate effects (exogenous). Covariates can be actor-dependent or dyad-dependent.

Convenient definition of evaluation function is a weighted sum

\[
f_i(\beta, x) = \sum_{k=1}^{L} \beta_k s_{ik}(x),
\]

where the weights \( \beta_k \) are statistical parameters indicating strength of effect \( s_{ik}(x) \).
Longitudinal modeling of social networks

Choose possible network effects for actor $i$, e.g.:
(others to whom actor $i$ is tied are called here $i$’s ‘friends’)

1. **out-degree effect**, controlling the density / average degree,
   \[ s_{i1}(x) = x_{i+} = \sum_j x_{ij} \]

2. **reciprocity effect**, number of reciprocated ties
   \[ s_{i2}(x) = \sum_j x_{ij} x_{ji} \]

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Four potential effects representing network closure:

3. **transitive triplets effect**, number of transitive patterns in $i$’s ties
   \((i \rightarrow j, j \rightarrow h, i \rightarrow h)\)
   \[ s_{i3}(x) = \sum_{j,h} x_{ij} x_{jh} x_{ih} \]

4. **transitive ties effect**, number of actors $j$ to whom $i$ is tied indirectly
   (through at least one intermediary: $x_{ih} = x_{hj} = 1$)
   and also directly $x_{ij} = 1$,
   \[ s_{i4}(x) = \#\{j \mid x_{ij} = 1, \max_h (x_{ih} x_{hj}) > 0\} \]
5. *indirect ties effect*,

number of actors $j$ to whom $i$ is tied indirectly
(through at least one intermediary: $x_{ih} = x_{hj} = 1$)
but not directly $x_{ij} = 0$),

$= \text{number of geodesic distances equal to 2,}$

$s_i5(x) = \# \{ j \mid x_{ij} = 0, \max_h (x_{ih} x_{hj}) > 0 \}$

6. *balance* or structural equivalence,

similarity between outgoing ties of $i$
with outgoing ties of his friends,

$s_i6(x) = \sum_{j=1}^{n} x_{ij} \sum_{h=1}^{g} \left( 1 - \left| x_{ih} - x_{jh} \right| \right)$,

[note that $(1 - \left| x_{ih} - x_{jh} \right|) = 1$ if $x_{ih} = x_{jh},$
and 0 if $x_{ih} \neq x_{jh}$, so that

$\sum_{h=1}^{g} (1 - \left| x_{ih} - x_{jh} \right|)$

measures agreement between $i$ and $j$. ]
Differences between these three network closure effects:

- **transitive triplets effect:** $i$ more attracted to $j$ if there are *more* indirect ties $i \rightarrow h \rightarrow j$;
- **transitive ties effect:** $i$ more attracted to $j$ if there is *at least one* such indirect connection;
- **balance effect:** $i$ prefers others $j$ who make same choices as $i$.

Non-formalized theories usually do not distinguish between these different closure effects.

It is possible to 'let the data speak for themselves' and see what is the best formal representation of closure effects.

7. **in-degree related popularity effect**, sum friends’ in-degrees
   $$s_{17}(x) = \sum_j x_{ij} \sqrt{x_{i+j}} = \sum_j x_{ij} \sqrt{\sum_h x_{jh}}$$
   related to dispersion of in-degrees
   (can also be defined without the $\sqrt{}$ sign);

8. **out-degree related popularity effect**, sum friends’ out-degrees
   $$s_{18}(x) = \sum_j x_{ij} \sqrt{x_{j+i}} = \sum_j x_{ij} \sqrt{\sum_h x_{jh}}$$
   related to association in-degrees — out-degrees;

9. **Outdegree-related activity effect**,
   $$s_{19}(x) = \sum_j x_{ij} \sqrt{x_{i+j}} = x_{i+}^{1.5}$$
   related to dispersion of out-degrees;

10. **Indegree-related activity effect**,
    $$s_{110}(x) = \sum_j x_{ij} \sqrt{x_{i+j}} = x_{i+} \sqrt{x_{i+j}}$$
    related to association in-degrees — out-degrees;
11. **three-cycle effect**, number of three-cycles in \(i\)'s ties
\[ (i \rightarrow j, \ j \rightarrow h, \ h \rightarrow i) \]
\[ s_{i11}(x) = \sum_{j,h} x_{ij} x_{jh} x_{hi} \]

This represents a kind of generalized reciprocity, and absence of hierarchy.

12. ... and potentially many others ...

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**Assortativity effects:**

Preferences of actors dependent on their degrees. Depending on their own out- and in-degrees, actors can have differential preferences for ties to others with also high or low out- and in-degrees.

Together this yields 4 possibilities:

- out ego - out alter degrees
- out ego - in alter degrees
- in ego - out alter degrees
- in ego - in alter degrees

All these are product interactions between the two degrees. Here also the degrees could be replaced by their square roots.
Four kinds of evaluation function effect associated with actor covariate $v_i$.

This applies also to behavior variables $Z_h$.

13. **covariate-related popularity**, ‘alter’
   sum of covariate over all of $i$’s friends
   $s_{i13}(x) = \sum_j x_{ij} v_j$;

14. **covariate-related activity**, ‘ego’
   $i$’s out-degree weighted by covariate
   $s_{i14}(x) = v_i x_{i+}$;

15. **covariate-related similarity**, sum of measure of covariate similarity between $i$ and his friends,
   $s_{i15}(x) = \sum_j x_{ij} \text{sim}(v_i, v_j)$
   where $\text{sim}(v_i, v_j)$ is the similarity between $v_i$ and $v_j$,
   \[
   \text{sim}(v_i, v_j) = 1 - \frac{|v_i - v_j|}{R_V},
   \]
   $R_V$ being the range of $V$;

16. **covariate-related interaction**, ‘ego × alter’
   $s_{i16}(x) = v_i \sum_j x_{ij} v_j$;
Evaluation function effect for dyadic covariate $w_{ij}$:

17. **covariate-related preference**, sum of covariate over all of $i$’s friends, i.e., values of $w_{ij}$ summed over all others to whom $i$ is tied, 

$$s_{17}(x) = \sum_j x_{ij} w_{ij}.$$ 

If this has a positive effect, then the value of a tie $i \rightarrow j$ becomes higher when $w_{ij}$ becomes higher.

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**Example**

Data collected by Gerhard van de Bunt: group of 32 university freshmen, 24 female and 8 male students.

Three observations used here ($t_1$, $t_2$, $t_3$): at 6, 9, and 12 weeks after the start of the university year. The relation is defined as a ‘friendly relation’.

Missing entries $x_{ij}(t_m)$ set to 0 and not used in calculations of statistics.

Densities increase from 0.15 at $t_1$ via 0.18 to 0.22 at $t_3$. 
Very simple model: only out-degree and reciprocity effects

<table>
<thead>
<tr>
<th>Effect</th>
<th>par. (s.e.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate $t_1 - t_2$</td>
<td>3.51 (0.54)</td>
</tr>
<tr>
<td>Rate $t_2 - t_3$</td>
<td>3.09 (0.49)</td>
</tr>
<tr>
<td>Out-degree</td>
<td>-1.10 (0.15)</td>
</tr>
<tr>
<td>Reciprocity</td>
<td>1.79 (0.27)</td>
</tr>
</tbody>
</table>

rate parameters:
per actor about 3 opportunities for change between observations;

out-degree parameter negative:
on average, cost of friendship ties higher than their benefits;

reciprocity effect strong and highly significant ($t = 1.79/0.27 = 6.6$).

Evaluation function is
\[
f_i(x) = \sum_j \left( -1.10 x_{ij} + 1.79 x_{ij} x_{ji} \right).
\]

This expresses ‘how much actor $i$ likes the network’.

Adding a reciprocated tie (i.e., for which $x_{ji} = 1$) gives
\[-1.10 + 1.79 = 0.69.\]

Adding a non-reciprocated tie (i.e., for which $x_{ji} = 0$) gives
\[-1.10,\]

i.e., this has negative benefits.

Gumbel distributed disturbances are added:
these have variance $\pi^2/6 = 1.645$ and s.d. 1.28.
Conclusion: reciprocated ties are valued positively, unreciprocated ties negatively; actors will be reluctant to form unreciprocated ties; by ‘chance’ (the random term), such ties will be formed nevertheless and these are the stuff on the basis of which reciprocation by others can start.

(Incoming unreciprocated ties, $x_{ji} = 1, x_{ij} = 0$ do not play a role because for the objective function only those parts of the network are relevant that are under control of the actor, so terms not depending on the outgoing relations of the actor are irrelevant.)

For an interpretation, consider the simple model with only the transitive ties network closure effect. The estimates are:

\[
\begin{array}{l|ll}
\text{Effect} & \text{Model 3} \\
\hline
\text{Rate } t_1 - t_2 & 3.89 \ (0.60) \\
\text{Rate } t_2 - t_3 & 3.06 \ (0.47) \\
\text{Out-degree} & -2.14 \ (0.38) \\
\text{Reciprocity} & 1.55 \ (0.28) \\
\text{Transitive ties} & 1.30 \ (0.41) \\
\end{array}
\]
Example: Personal network of ego.

\[ f_i(x) = \sum_j \left( -2.14x_{ij} + 1.55x_{ij}x_{ji} + 1.30x_{ij} \max_h (x_{ih}x_{hj}) \right) \]

(note: \( \sum_j x_{ij} \max_h (x_{ih}x_{hj}) \) is \#trans. ties)

so its current value for this actor is

\[ f_i(x) = -2.14 \times 4 + 1.55 \times 2 + 1.30 \times 3 = -1.56. \]
Options when ‘ego’ has opportunity for change:

<table>
<thead>
<tr>
<th>Options</th>
<th>out-degr</th>
<th>recipr</th>
<th>trans. ties</th>
<th>gain</th>
<th>prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>0.00</td>
<td>0.061</td>
</tr>
<tr>
<td>New tie to C</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>+2.01</td>
<td>0.455</td>
</tr>
<tr>
<td>New tie to D</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>+0.46</td>
<td>0.096</td>
</tr>
<tr>
<td>New tie to G</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>+0.46</td>
<td>0.096</td>
</tr>
<tr>
<td>Drop tie to A</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>−3.31</td>
<td>0.002</td>
</tr>
<tr>
<td>Drop tie to B</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>−0.46</td>
<td>0.038</td>
</tr>
<tr>
<td>Drop tie to E</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>+0.84</td>
<td>0.141</td>
</tr>
<tr>
<td>Drop tie to F</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>+0.59</td>
<td>0.110</td>
</tr>
</tbody>
</table>

The actor adds random influences to the gain (with s.d. 1.28), and chooses the change with the highest total ‘value’.

Model with more structural effects

<table>
<thead>
<tr>
<th>Model 3</th>
<th>par.</th>
<th>(s.e.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rate $t_1 - t_2$</td>
<td>4.64</td>
<td>(0.80)</td>
</tr>
<tr>
<td>Rate $t_2 - t_3$</td>
<td>3.53</td>
<td>(0.57)</td>
</tr>
<tr>
<td>Out-degree</td>
<td>−0.90</td>
<td>(0.58)</td>
</tr>
<tr>
<td>Reciprocity</td>
<td>2.27</td>
<td>(0.41)</td>
</tr>
<tr>
<td>Transitive triplets</td>
<td>0.35</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Transitive ties</td>
<td>0.75</td>
<td>(0.45)</td>
</tr>
<tr>
<td>Three-cycles</td>
<td>−0.72</td>
<td>(0.21)</td>
</tr>
<tr>
<td>In-degree popularity ($\sqrt{\cdot}$)</td>
<td>−0.71</td>
<td>(0.27)</td>
</tr>
</tbody>
</table>

Conclusions:
Reciprocity, transitivity; negative 3-cycle effect; negative popularity effect.
### Add effects of gender & program, smoking similarity

<table>
<thead>
<tr>
<th>Effect</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate $t_1 - t_2$</td>
<td>4.71 (0.80)</td>
</tr>
<tr>
<td>Rate $t_2 - t_3$</td>
<td>3.54 (0.59)</td>
</tr>
<tr>
<td>Out-degree</td>
<td>-0.81 (0.61)</td>
</tr>
<tr>
<td>Reciprocity</td>
<td>2.14 (0.45)</td>
</tr>
<tr>
<td>Transitive triplets</td>
<td>0.33 (0.06)</td>
</tr>
<tr>
<td>Transitive ties</td>
<td>0.67 (0.46)</td>
</tr>
<tr>
<td>Three-cycles</td>
<td>-0.64 (0.22)</td>
</tr>
<tr>
<td>In-degree popularity ($\sqrt{\cdot}$)</td>
<td>-0.72 (0.28)</td>
</tr>
<tr>
<td>Sex (M) alter</td>
<td>0.52 (0.27)</td>
</tr>
<tr>
<td>Sex (M) ego</td>
<td>-0.15 (0.27)</td>
</tr>
<tr>
<td>Sex similarity</td>
<td>0.21 (0.22)</td>
</tr>
<tr>
<td>Program similarity</td>
<td>0.65 (0.26)</td>
</tr>
<tr>
<td>Smoking similarity</td>
<td>0.25 (0.18)</td>
</tr>
</tbody>
</table>

**Conclusions:**
Trans. ties now not needed any more to represent transitivity; men more popular; program similarity.

To interpret the three effects of actor covariate *gender*, it is more instructive to consider them simultaneously. Gender was coded originally by with 1 for $F$ and 2 for $M$. This dummy variable was centered (mean was subtracted) but this only adds a constant to the values presented next, and does not affect the differences between them.

Therefore we may do the calculations with $F = 0, \ M = 1$.  

---

The joint effect of the gender-related effects for the tie variable $x_{ij}$ from $i$ to $j$ is

$$-0.15 z_i + 0.52 z_j + 0.21 I\{z_i = z_j\}.$$

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>0.21</td>
<td>0.52</td>
</tr>
<tr>
<td>M</td>
<td>-0.15</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Conclusion:
men seem not to like female friends...?
Longitudinal modeling of social networks

With this extension, the relative log-probabilities are

\[ f_i(\beta, x(i \rightarrow j)) - x_{ij} g_i(\gamma, x, j). \]

(Note that \( x_{ij} \) is the indicator of the current tie, before the change.)

The endowment function again can be a weighted sum

\[ g_i(\gamma, x, j) = \sum_{h=1}^{H} \gamma_h r_{ijh}(x). \]

Examples of components of endowment function:

1. \( \gamma_1 x_{ji} \)
   \( \gamma_1 \) extra benefits of a reciprocated tie.

2. \( \gamma_2 w_{ij} \)
   effect of dyadic covariate \( w_{ij} \)
   different for creating than for breaking a tie.

3. ... all other effects used also in the evaluation function.
Add endowment effect of reciprocated tie

<table>
<thead>
<tr>
<th>Effect</th>
<th>Model 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>par.</td>
</tr>
<tr>
<td>Rate $t_1 - t_2$</td>
<td>5.45</td>
</tr>
<tr>
<td>Rate $t_2 - t_3$</td>
<td>4.05</td>
</tr>
<tr>
<td>Out-degree</td>
<td>-0.62</td>
</tr>
<tr>
<td>Reciprocity</td>
<td>1.39</td>
</tr>
<tr>
<td>Transitive triplets</td>
<td>0.38</td>
</tr>
<tr>
<td>Three-cycles</td>
<td>-0.60</td>
</tr>
<tr>
<td>In-degree popularity ($\sqrt{\cdot}$)</td>
<td>-0.70</td>
</tr>
<tr>
<td>Sex (M) alter</td>
<td>0.63</td>
</tr>
<tr>
<td>Sex (M) ego</td>
<td>-0.29</td>
</tr>
<tr>
<td>Sex similarity</td>
<td>0.29</td>
</tr>
<tr>
<td>Program similarity</td>
<td>0.78</td>
</tr>
<tr>
<td>Smoking similarity</td>
<td>0.34</td>
</tr>
<tr>
<td>Endowment reciprocated tie</td>
<td>2.18</td>
</tr>
</tbody>
</table>

Transitive ties effect omitted.

Evaluation effect reciprocity: 1.39
Endowment reciprocated tie: 2.18

The overall (combined) reciprocity effect was 2.14.
With the split between the evaluation and endowment effects, it appears now that the value of reciprocity for creating a tie is 1.39, and for withdrawing a tie $1.39 + 2.18 = 3.57$.
Thus, there is a very strong barrier against the dissolution of reciprocated ties.
Extended model specification

2. Non-constant rate function $\lambda_i(\alpha, x)$.

This means that some actors change their ties more quickly than others, depending on covariates or network position.

Dependence on covariates:

$$
\lambda_i(\alpha, x) = \rho_m \exp\left(\sum_h \alpha_h v_{hi}\right).
$$

$\rho_m$ is a period-dependent base rate.

(Rate function must be positive; $\Rightarrow$ exponential function.)

Dependence on network position:

e.g., dependence on out-degrees:

$$
\lambda_i(\alpha, x) = \exp(\alpha_1 x_{i+}).
$$

Also, in-degrees and $\#$ reciprocated ties of actor $i$ may be used.

Now the parameter is $\theta = (\rho, \alpha, \beta, \gamma)$. 
Continuation example

Rate function depends on out-degree:
those with higher out-degrees
also change their tie patterns more quickly.

endowment function depends on tie reciprocation

\[ g_i(\gamma, x, j) = \gamma_1 x_{ji} \]

Reciprocity operates differently
for tie initiation than for tie withdrawal.

Parameter estimates model with rate and endowment effects

<table>
<thead>
<tr>
<th>Effect</th>
<th>Model 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate (period 1)</td>
<td>3.99 (0.70)</td>
</tr>
<tr>
<td>Rate (period 2)</td>
<td>2.93 (0.48)</td>
</tr>
<tr>
<td>Out-degree effect on rate</td>
<td>0.041 (0.034)</td>
</tr>
<tr>
<td>Out-degree</td>
<td>-0.79 (0.57)</td>
</tr>
<tr>
<td>Reciprocity</td>
<td>1.51 (0.54)</td>
</tr>
<tr>
<td>Transitive triplets</td>
<td>0.35 (0.05)</td>
</tr>
<tr>
<td>Three-cycles</td>
<td>-0.57 (0.19)</td>
</tr>
<tr>
<td>In-degree popularity ( (\sqrt{\cdot}) )</td>
<td>-0.59 (0.27)</td>
</tr>
<tr>
<td>Gender ego</td>
<td>-0.33 (0.31)</td>
</tr>
<tr>
<td>Gender alter</td>
<td>0.57 (0.27)</td>
</tr>
<tr>
<td>Gender similarity</td>
<td>0.30 (0.24)</td>
</tr>
<tr>
<td>Program similarity</td>
<td>0.80 (0.26)</td>
</tr>
<tr>
<td>Smoking similarity</td>
<td>0.36 (0.19)</td>
</tr>
<tr>
<td>Endowment recipr. tie</td>
<td>1.82 (0.97)</td>
</tr>
</tbody>
</table>
**Conclusion:**

non-significant tendency that actors with higher out-degrees change their ties more often ($t = 0.041/0.034 = 1.2$),

value of reciprocation is larger for termination of ties than for creation ($t = 1.82/0.97 = 1.88$).

---

**Non-directed networks**

The actor-driven modeling is less straightforward for non-directed relations, because two actors are involved in deciding about a tie.

Various modeling options are possible:

1. Forcing model:
   
one actor takes the initiative and unilaterally imposes that a tie is created or dissolved.
2. Unilateral initiative with reciprocal confirmation:
   one actor takes the initiative and proposes a new tie
   or dissolves an existing tie;
   if the actor proposes a new tie, the other has to confirm,
   otherwise the tie is not created.

3. Pairwise conjunctive model:
   a pair of actors is chosen and reconsider whether a tie
   will exist between them; a new tie is formed if both agree.

4. Pairwise disjunctive (forcing) model:
   a pair of actors is chosen and reconsider whether a tie
   will exist between them;
   a new tie is formed if at least one wishes this.

5. Pairwise compensatory (additive) model:
   a pair of actors is chosen and reconsider whether a tie
   will exist between them; this is based
   on the sum of their utilities for the existence of this tie.

Option 1 is close to the actor-driven model for directed relations.

In options 3–5, the pair of actors \((i, j)\) is chosen
depending on the product of the rate functions \(\lambda_i \lambda_j\)
(under the constraint that \(i \neq j\)).

The numerical interpretation of the ratio function
differs between options 1–2 compared to 3–5.

The decision about the tie is taken on the basis of the objective
functions \(f_i f_j\) of both actors.
2. Estimation

Suppose that at least 2 observations on $X(t)$ are available, for observation moments $t_1, t_2$.
(Extension to more than 2 observations is straightforward.)

*How to estimate $\theta$?*

*Condition on $X(t_1)$:*
the first observation is accepted as given, contains in itself no observation about $\theta$.

*No assumption of a stationary network distribution.*

Thus, simulations start with $X(t_1)$.

2A. Method of moments

Choose a suitable statistic $Z = (Z_1, \ldots, Z_K)$, i.e., $K$ variables which can be calculated from the network; the statistic $Z$ must be *sensitive* to the parameter $\theta$ in the sense that higher values of $\theta_k$ lead to higher values of the expected value $E_{\hat{\theta}}(Z_k)$;

determine value $\hat{\theta}$ of $\theta = (\rho, \beta)$ for which observed and expected values of suitable $Z$ statistic are equal:

$$E_{\hat{\theta}} \{ Z \} = z.$$
Questions:

- What is a suitable \((K\text{-dimensional})\) statistic?
  
  Corresponds to objective function.

- How to find this value of \(\theta\)?
  
  By stochastic approximation (Robbins-Monro process) based on repeated simulations of the dynamic process, with parameter values getting closer and closer to the moment estimates.

Suitable statistics for method of moments

Assume first that \(\lambda_i(x) = \rho = \theta_1\), and 2 observation moments.

This parameter determines the expected “amount of change”.

A sensitive statistic for \(\theta_1 = \rho\) is

\[
C = \sum_{i,j=1}^{g} \sum_{i \neq j} |X_{ij}(t_2) - X_{ij}(t_1)|,
\]

the “observed total amount of change”.
For the weights $\beta_k$ in the evaluation function

$$f_i(\beta, x) = \sum_{k=1}^{L} \beta_k s_{ik}(x),$$

a higher value of $\beta_k$ means that all actors strive more strongly after a high value of $s_{ik}(x)$, so $s_{ik}(x)$ will tend to be higher for all $i, k$.

This leads to the statistic

$$S_k = \sum_{i=1}^{n} s_{ik}(X(t_2)).$$

This statistic will be sensitive to $\beta_k$:
a high $\beta_k$ will to lead to high values of $S_k$.

Moment estimation will be based on the vector of statistics

$$Z = (C, S_1, ..., S_{K-1}).$$

Denote by $z$ the observed value for $Z$.
The moment estimate $\hat{\theta}$ is defined as the parameter value for which the expected value of the statistic is equal to the observed value:

$$E_{\hat{\theta}} \{Z\} = z.$$
Robbins-Monro algorithm

The moment equation $E_{\hat{\theta}}\{Z\} = z$ cannot be solved by analytical or the usual numerical procedures, because $E_{\theta}\{Z\}$ cannot be calculated explicitly.

However, the solution can be approximated by the Robbins-Monro (1951) method for stochastic approximation.

Iteration step:

$$\hat{\theta}_{N+1} = \hat{\theta}_N - a_N D^{-1}(z_N - z),$$

where $z_N$ is a simulation of $Z$ with parameter $\hat{\theta}_N$, $D$ is a suitable matrix, and $a_N \rightarrow 0$.

Covariance matrix

The method of moments yields the covariance matrix

$$\text{cov}(\hat{\theta}) \approx D_{\theta}^{-1} \Sigma_{\theta} D_{\theta}^{-1}$$

where

$$\Sigma_{\theta} = \text{cov}\{Z|X(t_1) = x(t_1)\}$$

$$D_{\theta} = \frac{\partial}{\partial \theta} E\{Z|X(t_1) = x(t_1)\}.$$
After the presumed convergence of the algorithm for approximately solving the moment equation, extra simulations are carried out

(a) to check that indeed $E_{\hat{\theta}}\{Z\} \approx z$ ,
(b) to estimate $\Sigma_{\theta}$,
(c) and to estimate $D_{\theta}$

using a score function algorithm
(earlier algorithm used difference quotients and common random numbers).

Modified estimation method:

*conditional estimation*.

Condition on the observed numbers of differences between successive observations,

\[ c_m = \sum_{i,j} | x_{ij}(t_{m+1}) - x_{ij}(t_m) |. \]
For continuing the simulations do not mind the values of the time variable $t$, but continue between $t_m$ and $t_{m+1}$ until the observed number of differences

$$\sum_{i,j} | X_{ij}(t) - x_{ij}(t_m) |$$

is equal to the observed $c_m$. This is defined as time moment $t_{m+1}$.

This procedure is a bit more stable; requires modified estimator of $\rho_m$.

Computer algorithm has 3 phases:

1. brief phase for preliminary estimation of $\partial E_\theta \{Z\}/\partial \theta$ for defining $D$;
2. estimation phase with Robbins-Monro updates, where $a_N$ remains constant in subphases and decreases between subphases;
3. final phase where $\theta$ remains constant at estimated value; this phase is for checking that

$$E_{\hat{\theta}} \{Z\} \approx z,$$

and for estimating $D_{\theta}$ and $\Sigma_{\theta}$ to calculate standard errors.
Extension: more periods

The estimation method can be extended to more than 2 repeated observations: observations \( x(t) \) for \( t = t_1, \ldots, t_M \).

Parameters remain the same in periods between observations except for the basic rate of change \( \rho \) which now is given by \( \rho_m \) for \( t_m \leq t < t_{m+1} \).

For the simulations, the simulated network \( X(t) \) is reset to the observation \( x(t_m) \) whenever the time parameter \( t \) passes the observation time \( t_m \).

The statistics for the method of moments are defined as sums of appropriate statistics calculated per period \( (t_m, t_{m+1}) \).

The procedures are implemented in the program

\textsc{Simulation Investigation for Empirical Network Analysis}

(current version is 3.2) which can be downloaded from

http://www.stats.ox.ac.uk/siena/;

(programmed by Tom Snijders, Christian Steglich, Ruth Ripley, Michael Schweinberger, Mark Huisman).
Currently, the entire Siena program is being programmed again as an R package, by Ruth Ripley and Krists Boitmanis.

This will be Siena 4, first beta release June 2009; the functionality will gradually be extended to offer everything that is now included within Siena 3.

Advantages: speed; open source; integration with R.