RPAD Welcome Week

Math Refresher
Unit II:
Exponents; Graphing
Stephen Weinberg
Who I Am

• Stephen Weinberg
• Economist
• MPA Courses:
  o RPAD 503 (core microeconomics course)
  o RPAD 643 (public expenditures)
  o RPAD 645 (psychological economics)
  o RPAD 654 (health policy)
  o RPAD 688 (STATA programming)
Agenda

• Overview
• Graphing
  o Slopes
  o Intercepts
  o Areas
• Exponents
  o Radicals
  o Logarithms
Overview

• There is a lot of math in an MPA
• Why?
  o Policy AND management are data driven
  o CANNOT evaluate evidence without some basic numeracy
  o In this program, math is crucial to success in RPAD 501 (budgets), 503 (economics), 504 (data), and 505 (statistics)
• There is a lot of math in the world
• Luckily, EVERYONE can do math
Overview

• Key to math: practice
• (Almost) nobody gets this stuff the first time
• If you don’t use it, you tend to forget it
• So we’re going to review some stuff you should have seen in high school
Overview

• Suggested book: Barron’s *Forgotten Algebra*

• On order at Mary Jane Books (see ad in Welcome Week booklet)

• Today: chapters 7, 12, 23, 24, 30, plus areas
Graphing

• Complete change of subject
• One of the most powerful tools of mathematics is the GRAPH
• Powerful means of summarizing relationships between two variables
• You MUST be able to graph linear equations
Graphing

• We graph in “Cartesian” coordinates (named after Descartes)
• Two axes, the horizontal and the vertical
  o Usually called X and Y, but be flexible
  o In 503, for example, we’ll usually graph Q and P
• The axes meet at the ORIGIN
Graphing

• A coordinate on a graph is a pair of numbers, such as (x,y) or (Q,P)
• The first number is the HORIZONTAL position
• The second number is the VERTICAL position
• Example: graph (2,3), (5,-4), and (0,7)
Graphing

• Terminology: when we draw a point on a graph, we call it “plotting” the point
Graphing

• Aside: some students like to be very neat when drawing graphs, and break out rulers and evenly measure the distances
• This is NOT a good use of time on a test
• Just sketch it reasonably
Graphing

• A LINE shows a relationship between two variables
• A line shows, for every x value, what y value corresponds to it
• Thus, a line consists of one equation with two variables
Graphing

• You may have seen many forms for a line in high school
• My preferred form is
  \[ Y = mX + b \]
• That is, I prefer to write lines so that
  o The vertical-axis variable is all by itself on the left-hand side
  o The horizontal-axis variable has been multiplied by something and added to something
Graphing

• $Y = mX + b$
  
  o $b$ is the “$Y$-intercept”
    • It tells us what value $Y$ has when $X=0$
  
  o $m$ is the “slope”
    • Slope: Change in $Y$ over Change in $X$
    • How much does $Y$ change by when $X$ changes by 1 unit?
Graphing

• To graph a linear equation
  \[ Y = mX + b \]
  o Draw and label your axes
  o Plot the Y-intercept (b)
  o Move over 1 unit in X and m units in Y and plot a second point
  o Draw a line through these two points
Graphing

• For each equation, find the Y-intercept and slope, and graph the line.

1. $Y = 25X + 10$
2. $P = 5 - 0.2Q$
3. $H = -8J - 2$
4. $Fred = 0.5Wilma - 4$
5. $Y = 10$
Graphing

- Note: the greater the slope, the steeper the line
- A slope of 0 means a line is completely flat
- A slope of $\infty$ means a line is completely vertical
- A positive slope means a line is upward sloping
- A negative slope means a line is downward sloping
Graphing

• Note Well: you may not be given a line in this form
• You may need to solve the equation to get it into the correct form
• Example: $2X + 3Y = 9$
  • $3Y = 9 - 2X$
  • $Y = 3 - \frac{2}{3} X$
Graphing

• The SLOPE can be used to quickly estimate the change in Y for any change in X
  • Example: $Y = 5 - 4X$
  • If X increases by 2 units, Y DECREASES by 8 units
Graphing

• P = 100 – 0.25Q
  o If Q goes up by 8 units, what happens to P?
• A = -5 + 4B
  o If B goes DOWN by 3 units, what happens to A?
• A = 5
  o If B goes UP by 1 unit, what happens to A?
• Y = 12 + 2X
  o If Y goes UP by 3 units, what happens to X?
Graphing

• $P = 100 - 0.25Q$
  - If $Q$ goes up by 8 units, $P$ goes down 2
• $A = -5 + 4B$
  - If $B$ goes DOWN by 3 units, $A$ goes down 12
• $A = 5$
  - If $B$ goes UP by 1 unit, $A$ stays the same
Graphing

• $Y = 12 + 2X$
  o If $Y$ goes UP by 3 units, what happens to $X$?
  o WARNING: this equation is NOT in the right form to answer this question!!!!!
  o Slope = change in $Y$ / change in $X = 2$
    • This tells us how much $Y$ changes if we change $X$ by 1 unit
  o If we want change in $X$ / change in $Y$, we need to INVERT the slope
  o $X$ changes by $\frac{1}{2}$ when $Y$ changes by 1
  o $X$ changes by $\frac{3}{2}$ when $Y$ increases by 1
  o If you don’t see this, try re-solving the equation for $X$ in terms of $Y$
Areas

• In 503, we very often need to calculate the areas of regions we’ve graphed
  o Area of a Rectangle: \( b \times h \)
  o Area of a Triangle: \( 0.5b \times h \)
Areas

• What is the height of a triangle?
  o The distance from any corner to the opposite side, if it hits the opposite side at a RIGHT ANGLE
    • The base is then the length of that side
  o Very easy to see in a Right Triangle
  o Example: area between X-axis, Y-axis, and line $Y = 10 - 0.5X$
Areas

• Example: area between X-axis, Y-axis, and line \( Y = 10 - 0.5X \)

• We need to know the distances from the origin to the Y-int and the X-int

• Y-intercept: \((0,10)\) (read right off formula)

• X-int: \(0 = 10 - 0.5X; 10 = 0.5X; X = 20\)
  \(\circ X\)-int = \((20,0)\)
Areas

• TWO CHOICES for what the “height” is
  o The height could be the line from (0,10) to (0,0). The base would be the line from (0,0) to (20,0). Area = 0.5*20*10 = 100
  o The height could be the line from (20,0) to (0,0). The base would be the line from (0,0) to (0,10). Area = 0.5*10*20 = 100
Areas

• Examples: Find the area between
  o X-axis, Y-axis, and line $Y = 8 - X$
  o X-axis, Y-axis, and line $Y = -5 + 0.2X$
  o $Y = 4$, Y-axis, and $Y = 20 - 2X$
Areas

• Examples: Find the area between
  o X-axis, Y-axis, and line $Y = 8 - X$
    • Y-int = (0,8); X-int = (8,0)
    • Area = $0.5 \times 8 \times 8 = 32$
  o X-axis, Y-axis, and line $Y = -5 + 0.2X$
    • Y-int = (0,-5); X-int = (25,0)
    • Area = $0.5 \times 5 \times 25 = 125/2 = 62.5$
Areas

• Y = 4, Y-axis, and Y = 20 – 2X
  o We need the distance between (0,4) and the Y-intercept and between (0,4) and the point on the line where Y=4
  o Y-intercept = (0,20)
  o Point where Y=4: 4 = 20 – 2X; X = 8
  o Area = 0.5*(20-4)*(8-0) = 64
Areas

• Trickier: area between
  o Y-axis
  o Line D: $Y = 10 - X$
  o Line S: $Y = 4 + 0.5X$

• Refresher 3 will review how to solve for intersection of 2 lines

• Assert: they intersect at point (4,6)

• What is the area?
Areas

• Trickier: area between
  o Y-axis, $Y = 10 - X$, $Y = 4 + 0.5X$
  o Intersection at (4,6)
• What is the height?
• NOTE: draw a line from (4,6) to Y-axis
  o This line has length 4; that’s your “height”
    • (Yes, we’re taking the “height” horizontally. The triangle is sleepy, so it’s lying down.)
  o The base is the length of the side along the Y-axis, which is $10 - 4 = 6$
  o Area = $0.5 \times 6 \times 4 = 12$
Areas

• What is the area formed by
  o Y-axis
  o $Y = 20 - 0.5X$
  o $Y = 5 + 2.5X$
  o Intersection at (5,7.5)
Areas

• What is the area formed by
  - Y-axis
  - Y = 20 – 0.5X
  - Y = 5 + 2.5X
  - Intersection at (5,7.5)
  - Height is line from (5,7.5) to (0,7.5)
    • Length = 5
  - Base is line from (0,5) to (0,20)
    • Length = 15
  - Area = 0.5*5*15 = 75/2 = 37.5
Exponents

• Exponents are a notation for doing multiplication over and over and over again
  • $2^4$ means “multiply 2 by itself 4 times”
  • $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
• We call this “raising 2 to the power of 4”
Exponents

• Special terms:
  o $x^2$ is “x squared”
  o $x^3$ is “x cubed”
  o The thing being multiplied is the BASE
  o The number of times you multiply is the POWER or the exponent

• Special rule:
  o $x^0 = 1$, as long as x isn’t 0
Exponents

• What is
  o $4^3$
  o $2^5$
  o $1^{24}$
  o $(-1)^3$
  o $(-5)^2$
Exponents

• Solutions:
  o $4^3 = 4 \cdot 4 \cdot 4 = 64$
  o $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$
  o $1^{24} = 1$
  o $(-1)^3 = -1$
  o $(-5)^2 = 25$
Exponents

• Things you CANNOT do with exponents:
  o ADD terms with different powers
    • $x^2 + x^5 \neq x^7$
  o ADD terms with different bases
    • $4^3 + 6^3 \neq 10^3$
  o Basically, addition and exponents do not mix
  o The best you can do is add them together when the BASE and the EXPONENT are both the same
    • $4x^3 + 6x^3 = 10x^3$
What CAN you do with exponents?

• Here are some basic rules that often come in handy (see p51)
  
  $x^2 \cdot x^4 = x^{2+4} = x^6$
  
  $(x^2)^4 = x^8$
  
  $(x \cdot y)^3 = x^3 \cdot y^3$
  
  $(x/y)^5 = x^5 / y^5$
What CAN you do with exponents?

• Why?
  o Rewrite the exponents as a bunch of multiplications, and then re-group terms

• $x^2 \cdot x^4 = (x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = (x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^6$
Simplifying

- You can use these rules to simplify expressions
- Simplifying is a useful step towards solving an equation
- Example:
  - $x^2 \cdot x^2 = 16$ is much easier to solve if you can rewrite it to $x^4 = 16$
Practice Problems

Simplify:
(That is, rewrite without any parentheses, and with each variable used as few times as possible)

1. \((5y)^2\)
2. \(4Q^0\)
3. \(x^2 \cdot (x^5)^2\)
4. \((-3c)^2\)
5. \(2P^3 + 3P^4\)
Practice Problems

Simplify:

1. \((5y)^2 = 25y^2\)
2. \(4Q^0 = 4 \cdot 1 = 4\)
3. \(x^2 \cdot (x^5)^2 = x^2 \cdot x^{10} = x^{12}\)
4. \((-3c)^2 = 9c^2\)
5. \(2P^3 + 3P^4 = 2P^3 + 3P^4\)
Exponents

• We’ve been looking at mathematical statements like

\[ b^m = x \]

• So far, we’ve taken b and m and found x

• Two other processes:
  o Radicals/Roots: Given x and m, find b
  o Logarithms: Given x and b, find m
Exponents

- Example: $2^3 = 8$
- “Two cubed is 8”
- “The cube root of 8 is 2”
- “The log-base-2 of 8 is 3”

Roots: $\sqrt[3]{8} = 2$

Logarithms: $\log_2 8 = 3$
Roots

• Even-powered roots have two possible solutions: the positive or the negative
• The square root of 25 could be either 5 or -5
• We call the positive square root the “principal square root”
Roots

• Negative numbers do NOT have even-powered roots
• The square root of -16 is NOT a real number
• (There’s a whole system of math, called “complex algebra,” that uses “imaginary numbers” based on the square root of negative numbers. This has proved surprisingly useful in many settings. Thankfully, this is NOT one of those settings.)
Roots

• It turns out that roots follow the EXACT same rules as exponents
• Actually, roots are a special case of exponents
• We can re-write roots as fractional exponents
• \( \sqrt{25} = 25^{1/2} \)
Things You Can Do with Roots

\[ \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \]

\[ (ab)^{1/2} = a^{1/2} \cdot b^{1/2} \]

\[ \sqrt{x^2 y^3} = \pm xy \sqrt{y} \]

\[ \sqrt{3} + \sqrt{2} = \sqrt{3} + \sqrt{2} \]
Problems

• Simplify:

1. $\sqrt{ab} \cdot \sqrt{a}$
2. $\sqrt{2} \cdot \sqrt{10}$
3. $\sqrt{x^2}$
4. $\sqrt[3]{15} \cdot \sqrt[3]{15} \cdot \sqrt[3]{15}$
5. $\sqrt{x} + \frac{3}{\sqrt{x}}$
Problems

• Simplify:

1. \( \sqrt{ab} \cdot \sqrt{a} = \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{a} = \sqrt{a^2} \cdot \sqrt{b} = a\sqrt{b} \)

2. \( \sqrt{2} \cdot \sqrt{10} = \sqrt{20} \)

   It may or may not be convenient to rewrite this as

   \( \sqrt{4 \cdot 5} = 2\sqrt{5} \)

3. \( \sqrt{x^2} = \pm x \)

4. \( 3\sqrt{15} \cdot 3\sqrt{15} \cdot 3\sqrt{15} = (3\sqrt{15})^3 = (15^{1/3})^3 = 15^1 = 15 \)

5. \( \sqrt{x} + 3\sqrt{x} = \sqrt{x} + 3\sqrt{x} \)
Logs

• Logs are one of the least intuitive subjects in algebra
• Unfortunately, they’re very important
• My goal here is to remind you that they exist and that they have some (incredibly unintuitive) rules
Logs

- $\log_b N = x$
- The log-base-$b$ of $N$ is $x$
- What power do you need to raise $b$ to, to get $N$?
- $4^3 = 64$, so $\log_4 64 = 3$
Logs

• Problems:
1. \( \log_2 64 \)
2. \( \log_5 125 \)
3. \( \log_{90} 1 \)
4. \( \log_8 2 \)
Logs

• Problems:
  1. $\log_2 64 = 6$
  2. $\log_5 125 = 3$
  3. $\log_{90} 1 = 0$
  4. $\log_8 2 = 1/3$
Logs

• Natural log: $\ln$
• It turns out that there’s a number, Euler’s constant, that we call $e$, that is dang useful to use with logs
• We call log-base-$e$ the “natural log”
• $\ln x$: “What power do we need to take $e$ to, to get $x$?”
• $e$ is like pi, in that it has an infinite number of decimals. It’s about $2.72$
Logs

• In always refers to log-base-e
• log, by itself, might mean log-base-e or log-base-10, depending on the writer
What Can You do with Logs?

• Logs have really weird rules
• I’m just going to point out that they exist
• We’re not going to practice them here
Log Rules

\[ \log_b AC = \log_b A + \log_b C \]

\[ \log_b \frac{A}{C} = \log_b A - \log_b C \]

\[ \log_b A^k = k \log_b A \]
Example of Logs Making Life Easier

• Economists often model an economy’s output using the Cobb-Douglass form:

\[ Y = AK^\alpha L^\beta \]

where \( Y \) is output, \( A \) is productivity, \( K \) is capital, and \( L \) is labor. \( \alpha \) and \( \beta \) are parameters that show how much capital and labor contribute to production.
Example of Logs Making Life Easier

\[ Y = AK^\alpha L^\beta \]

• Suppose you had data on output (Y), capital (K), and labor (L), and wanted to estimate A, \( \alpha \), and \( \beta \)
• Statistically, this would be a real @%@#$%
• It’s very hard to estimate multiplicative models in statistics
Example of Logs Making Life Easier

\[
\ln Y = \ln A + \alpha \ln K + \beta \ln L
\]

- If you log both sides, you can use log laws to change the multiplicative model into an additive one
- This is a trivially easy model to estimate statistically
- Yay logs!