1. A particle of mass $m$ is launched horizontally with velocity $v_0$ from the top of a cliff. As shown in the figure the origin is placed at the top of the cliff, the $x$ axis is horizontal, the $y$ axis is vertical, and the $z$ axis points out of the page.

(a) Find $x(t)$ and $y(t)$.

(b) Find the angular momentum about the $z$ axis as a function of time.

(c) What is the torque of the force of gravity about the $z$ axis as a function of time.

(d) Show that your answer to (c) is consistent with the result for (b).
2. A system consists of a linear chain of two particles of mass $m$ and three springs. The outer two springs have force constant $k$ and are attached to a particle on one side and to a wall on the other. The central spring connects the two particles and has force constant $4k$. Neglect transverse oscillations, that is, consider longitudinal oscillations along the chain only.

![Diagram of the system with particles and springs labeled]

(a) How many degrees of freedom are there? Find the Lagrangian and write down the equation of motion for each degree of freedom.

(b) Find the frequencies of the longitudinal normal modes.

(c) Briefly describe the normal modes.

(d) Just before $t = 0$ the system is at rest in its equilibrium position. At $t = 0$ the particle on the left is given a sudden impulse so that it acquires a speed $u$. Find the positions of the two particles as a function of time.
3. The orbits of a particle of mass $m$ moving around a black hole are described by the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{m}{2} \left( 1 - \frac{2k}{r} \right) \left( 1 + \frac{p_\phi^2}{m^2 r^2} \right)$$

where $(r, \phi)$ are polar coordinates, $(p_r, p_\phi)$ are the corresponding conjugate momenta, and $k$ is a positive constant.

(a) Write down Hamilton’s equations of motion.

(b) Which (if any) of the conjugate momenta are conserved? Is energy conserved? Justify your answers.

(c) Look at the effective potential for radial motion: Are circular orbits allowed? For what ranges of values $p_\phi$? Are they stable?
4. Electromagnetic waves are radiated by a point charge \( q \) that oscillates vertically above a \textbf{horizontal conducting plane} located at \( z = 0 \). The oscillations have amplitude \( a \) about a mean height \( h \) and frequency \( \omega \) so that the coordinates of the charge are

\[
x(t) = y(t) = 0 \quad \text{and} \quad z(t) = h + a \cos \omega t.
\]

Assume that both \( h \) and \( a \) are much smaller that the wavelength so that you may use the dipole approximation.

\( \text{(a)} \) Calculate the vector potential \( \vec{A}(\vec{r}, t) \) in the radiation zone.

\( \text{(b)} \) Calculate the radiated magnetic field \( \vec{B} \).

\( \text{(c)} \) Calculate the radiated electric field \( \vec{E} \).

\( \text{(d)} \) Calculate the (time averaged) angular distribution of radiated power, \( dP/d\Omega \).

Hints: In Gaussian units the vector potential radiated by an oscillating electric dipole \( \vec{p} e^{-i\omega t} \) at the origin is

\[
\vec{A}(\vec{r}, t) = -ik\vec{p} \frac{e^{ikr - i\omega t}}{r}
\]

where \( k = \omega/c \). You may need the identity \( \vec{\nabla} \times (\psi \vec{v}) = (\vec{\nabla} \psi) \times \vec{v} + \psi \vec{\nabla} \times \vec{v} \).
5. An electromagnetic plane wave propagates along the $+x$ axis in an inertial frame $O$. It is described by a vector potential

$$\vec{A}(\vec{r}, t) = a\vec{e}_z e^{-i\omega t + ikx} \quad \text{with} \quad \omega = ck$$

and the scalar potential $\phi$ vanishes.

(a) Calculate the electric and magnetic fields, $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$.

An observer $O'$ moves at a velocity $c/2$ relative to $O$ along the $x$ axis.

(b) Find the frequency $\omega'$ in the $O'$ frame. (Hint: $(\omega/c, \vec{k})$ is a four-vector.)

(c) Find the electric and magnetic fields in the $O'$ frame. (Hint: $(\phi, \vec{A})$ is a four-vector.)