Instruction:

Answer any four out of five questions

Please check below the four problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the four books at the end of the exam.

6. __________
7. __________
8. __________
9. __________
10. __________

Student Identification Code: ____________________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in May 2011.
6. In the presence of constant magnetic field $h$, the dynamics of a spin-1/2 particle placed in a spherical harmonic oscillator potential is governed by the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 + h S_z$$

Where $S_z$ is the z-component of the spin $S$ and $\omega > h > 0$.

a. Rewrite the Hamiltonian in terms of creation and annihilation operators

b. Find the full spectrum of the Hamiltonian; in particular, determine the energy of the ground state of the system.

c. The Hamiltonian is perturbed by $H' = \lambda \cdot S \cdot r$, where $S$ is the spin operator. Compute the shift of the ground state energy to second order in $\lambda$.

Hint: rewrite $r$ in terms of creation and annihilation operators.
7. Gravity-induced quantum interference, first observed in 1975 by Colella, Overhauser and Werner, can be described by considering a particle of mass $m$ in a gravitational potential $\phi_{grav}$ with a potential energy of $m\phi_{grav}$.

a. Recall that classically the force on the particle is the gradient of the potential. Write Newton’s second law and show that the dynamics of the particle does not depend on the mass.

b. Using this potential, write Schroedinger’s equation and show that rather than cancelling, the mass appears in combination with $\hbar$.

c. However, show using Ehrenfest’s Theorem that Newton’s second law is satisfied for expectation values, and consequently that the mass does not influence the dynamics of the expectation values. Do this by finding $\frac{d}{dt} < P >$. 

8. Consider an Ammonia molecule NH$_3$

The atom looks like a pyramid where the three Hydrogen atoms lie in a plane and create a potential barrier through which the Nitrogen atom can tunnel.

a. Given the potential well illustrated below, make two sketches illustrating the lowest energy symmetric wavefunction, $\phi^1_s(x)$, and lowest energy antisymmetric wavefunction, $\phi^1_a(x)$.

b. The energy of the antisymmetric wavefunction, $E^1_a$, is known to be greater than the energy of the symmetric wavefunction, $E^1_s$. Explain in words why this should be the case.

c. Assume that at time $t = 0$ the molecule is in the state

$$|\psi(x, t = 0) > = \frac{1}{\sqrt{2}} [ |\phi^1_s(x) > + |\phi^1_a(x) >].$$

Write $|\psi(x, t) >$.

d. Find the probability density $|\psi(x, t)|^2$ as well as the frequency with which the Nitrogen atom tunnels back-and-forth.
9. Calculate the first-order shift in the ground state of the hydrogen atom caused by the finite size of the proton. Assume that the proton is a uniformly-charged sphere of radius $r_0$. The ground state (point nucleus) wavefunction of the Hydrogen atom is

$$\psi_{100} = \left( \frac{1}{\pi a_0^3} \right)^{\frac{1}{2}} \exp \left( - \frac{r}{a_0} \right)$$

where $a_0$ is the Bohr radius.

Hint: Calculate the electric field inside the proton. From this, find the potential inside the proton. Treat the difference between the potential energy inside the finite-size proton and that for a point-charge proton as a perturbation.
10. A particle of mass $m$ and energy $E$ moves in a one-dimensional potential $V(x)$.

a. By writing the wavefunction $\psi(x) = \exp\left(\frac{iS(x)}{\hbar}\right)$ and expanding $S(x)$ in powers of $\hbar$, or otherwise, derive the WKB approximation

$$\psi(x) = \frac{A}{\sqrt{k(x)}} \exp\left(-\frac{i}{\hbar} \int k(x) dx\right)$$

with $k(x) = \sqrt{2m(V(x) - E)}$, for solutions to the time-independent Schroedinger equation in a region where $E < V(x)$, where $A$ is a constant.

b. State the conditions under which the approximation above is valid.

c. The particle described above tunnels through the barrier

$$V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 - Fx & x > 0 \end{cases}$$

where $V_0 > 0$ and $F > 0$, moving from left to right.

d. Make a sketch of $V(x)$ together with the general form of the wavefunction.

e. Using the WKB approximation, calculate the exponential factor in the tunneling probability through the barrier.

f. Describe the conditions on the parameters under which the WKB approximation is accurate in this case.