**Problem #1**

A. A projectile of mass m is shot vertically in the gravitational field. Its initial velocity is $v_0$. Assuming there is no air resistance, how high does m go?

B. Now assume the projectile is subject to a drag force $-cv^2$, where c is a positive constant and $v$ is the speed of the mass m. How high will this mass go?

C. Show your answer in part B agrees with your answer in part A in the limit that the constant $c \to 0$. 

Problem #2

Consider 4 masses m, all with charge +q located \((x,y) = (a,a), (a,-a), (-a,a)\) and \((-a,-a)\), as shown in the figure below.

A. What is the E field at the origin, \((x=0, y=0)\)?

B. What is the electric potential at the origin?

C. What is the potential energy stored in this system of four charges?

D. If these 4 masses are simultaneously released, what is the speed of each mass at large distance from the origin?

E. If these four charges rotate as a rigid body about the origin with angular velocity \(\omega\) what is the average current that results from the motion of these four charges. *Note, this current flows in a circle of radius \((\sqrt{2} \cdot a)\) centered at the origin.*

F. You can approximate the current as described in part E as continuous. What then is the magnetic field at the origin?
Problem #4

Description for part a:
A point charge q is placed a distance d above a semi-infinite grounded conductor as shown in the figure below.

a) What is the force on the charge q? (hint: the method of images is useful here)

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Description for parts b through f:
A point charge q is placed in a semi-infinite medium of permittivity \( \varepsilon_0 \) a distance d away from the plane interface (the x-y plane at z=0) that separates the above medium (z>0) from another semi-infinite dielectric medium (z<0) with permittivity \( \varepsilon \). See figure below.

b) Use the image method to find the potential \( \Phi(x,y,z) \) at an arbitrary point \((x,y,z)\) for \(z>0, z=0,\) and \(z<0\).

c) Find the electric displacement \( \mathbf{D} \) for \(z\leq0\)

d) Find the polarization \( \mathbf{P} = (\varepsilon-\varepsilon_0)\mathbf{E} \) of the medium for \(z\leq0\)

e) Find the polarization-surface-charge density \( \sigma_{pol} \) on the plane \(z=0\).

f) Find the force exerted on the charge q by the dielectric medium.
Problem #5

A flexible chain of total length L and total mass M hangs partially over the edge of a table, as shown in the figure. Initially, the chain is at rest with a length $x_0$ hanging over the edge.

A. Write the Lagrangian and Lagrange's equation in terms of the generalized coordinate $x$, as shown in the figure.

B. Solve for $x(t)$ using the initial conditions given above.

C. What is the generalized momentum, $p_x$, corresponding to the generalized coordinate $x$? Is $p_x$ a constant of the motion?

D. Write the Hamiltonian $H(x, p_x)$
Problem #6

A single particle with mass $m$ is placed in a cubic box with sides of length $L$. The potential $V(\vec{r})$ inside the box is zero. The box has rigid walls, so the particle’s wave function must vanish at the walls of the box.

![Cubical Box Diagram]

a) Use the time-independent Schrödinger equation to derive the energy levels of this particle.

Two different kinds of elementary particles can be placed in this same box with the same boundary conditions.

- Type A particles have mass $m$ and spin zero.
- Type B particles have mass $10m$ and spin $\frac{1}{2}$.

All type A particles are identical. All type B particles are identical. Assume there is no interaction potential energy between any of the particles.

b) Which particles (Type A or Type B) are fermions and which are bosons?
c) Which particles (Type A or Type B) are subject to the Pauli exclusion principle?

Three Type A particles are placed in the box.
d) Find the ground state energy this three-particle system.
e) Find the degeneracy of the ground state of part d).

Three Type B particles are placed in the box.
f) Find the ground state energy of this three-particle system.
g) Find the degeneracy of the ground state of part f).

Three Type A particles and three Type B particles are placed in the box, so there are six particles in the box.
h) Find the ground state energy of this six-particle system.
i) Find the degeneracy of the ground state of part h).
Problem #7

1. An electron with kinetic energy $10 \text{ eV}$ is scattered by a potential step of height $5 \text{ eV}$ at $x = 0$, as shown below. We can express the incident, reflected, and transmitted wave functions as $A_0 \exp(ik_1x)$, $A \exp(-ik_1x)$, and $B \exp(ik_2x)$, respectively, with $k_1 = (2mE)^{1/2}/\hbar$ and $k_2 = [2m(E-V)]^{1/2}/\hbar$.

(a) Use the boundary conditions that the wave function and the derivative of the wave function must be continuous at the step ($x = 0$), find the expression of the amplitudes $A$ and $B$ in terms of $A_0$.

(b) Calculate the numerical values of the reflection coefficient $R = |A/A_0|^2$ and transmission coefficient $T = 1 - R$.

(c) Calculate and draw the probability density $|\psi(x)|^2$ in the steady state on both sides of the step, assuming $A_0 = 1$.

(d) If the kinetic energy of the incident electron is $3 \text{ eV}$ instead of $10 \text{ eV}$, determine the reflection and transmission coefficients in this case.

![Graph](image-url)
Problem #8

Harmonic Oscillator in an E-Field

An electron is placed in a one-dimensional Harmonic oscillator with potential 
\[ V(x) = \frac{1}{2} kx^2 . \] 
At time \( t = 0 \) a constant electric field is turned on \( \vec{E} = E_0 \hat{x} \) in the x-direction.

a. Write down the Hamiltonian for time \( t > 0 \).

b. Use perturbation theory to find the Eigenvalue and Eigenfunction of the perturbed ground state to first order.

This problem can also be solved without perturbation theory by a change of variables. Do this in parts c and d below:

c. Find the Eigenvalues and Eigenfunctions of \( H \) in general (not just the ground state).

d. Find \( \langle x \rangle \) for all Eigenstates

Hint: The Eigenfunction of the \( n^{th} \) excited state of the unperturbed Hamiltonian is:

\[
\psi_n(x) = \left( \frac{m \omega}{\pi \hbar 2^n (n!)^2} \right)^{1/4} \exp \left( -\frac{m \omega x^2}{2\hbar} \right) H_n \left( \sqrt{\frac{m \omega}{\hbar}} \, x \right)
\]

where \( H_n(y) \) is the \( n \)th order Hermite polynomial, so that

\[
H_0(y) = 1 \\
H_1(y) = 2y \\
H_2(y) = 4y^2 - 2 \\
\]

…
Problem #9

A particle of mass $m$ moves along the $x$-axis. The potential energy is infinite for $x < 0$ so the wave function is non-zero only for positive $x$.

The particle is a solution to the time-independent Schrodinger equation in a potential $V(x)$ with $V(x) \to 0$ as $x \to \infty$. The wave function is

$$\psi(x) = Ax \exp(-kx)$$

a) Find a value for the constant $A$ so the wave function is normalized and

$$\int_{0}^{\infty} \psi^*(x)\psi(x)dx = 1$$

b) Use the Schrodinger equation to find the potential $V(x)$ for $x > 0$. Remember that $V(x) \to 0$ as $x \to \infty$.

c) Find the particle’s energy, $E$.

d) Find the expectation of the position, $\langle x \rangle = \int_{0}^{\infty} \psi^*(x)x\psi(x)dx$

e) Find the value of $x$ for which the potential energy $V(x)$ is equal to the energy $E$.

f) Obtain the integral that gives the probability that the particle can be found in the “forbidden region” where the potential energy $V(x)$ is greater that the energy $E$.

\[ \int_{0}^{\infty} x^n \exp(-x)dx = n! \]
Problem #10

The Hamiltonian for a harmonic oscillator can be expressed as

$$\hat{H}_0 = \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

where $[\hat{a}, \hat{a}^\dagger] = 1$. The energy eigenstates of this system are identified with the number states $|n\rangle$ on which $\hat{a}$ and $\hat{a}^\dagger$ act as

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

where

$$\langle m|n \rangle = \delta_{mn} \quad (m, n = 0, 2, 3, ...).$$

When the system is subjected to a time-dependent perturbation $\hat{H}'(t)$, it undergoes a transition from an initial state to other states. The probability amplitude of the transition from an initial state $|l\rangle$ to some other state $|k\rangle$ at time $t$ is given by

$$c_{kl}(t) = \frac{1}{i\hbar} \int_{-\infty}^{t} \langle k|\hat{H}'(t)|l\rangle e^{i\omega_{kl}t} dt$$

where

$$\omega_{kl} = (E_k^{(0)} - E_l^{(0)})/\hbar$$

and $E_n^{(0)}$ is an energy eigenvalue of the unperturbed system.

(a) Find the eigenvalues $E_n$ ($n = 0, 1, 2, ...$) of the unperturbed Hamiltonian $\hat{H}_0$.

(b) What are the energies (emitted or absorbed) to make a transition from the first excited state $|1\rangle$ to the ground states $|0\rangle$ and to the second excited state $|2\rangle$?

(c) Find the non-vanishing matrix elements $\langle k|\hat{a}^\dagger \hat{a}^3|1\rangle$ and their values.

(d) Suppose a time-dependent perturbation has the form

$$\hat{H}' = \begin{cases} 
0 & \text{if } t < 0 \\
K(\hat{a}^\dagger \hat{a}) e^{-\alpha t} \cos(\omega t) & \text{if } t > 0
\end{cases}$$

where $K$, $\alpha$ and $\omega$ are all constants. Suppose the oscillator is initially in the first excited states $|1\rangle$. Calculate the probability $P_{k1} = \lim_{t \to -\infty} |c_{k1}(t)|^2$ that the oscillator makes a transition from the initial state to the ground state $|0\rangle$ after a sufficiently long time. Obtain also the transition probability to the second excited state $|2\rangle$. 
Problem #11

The figure below shows a monatomic gas going through an Otto cycle. Hint: $PV^{5/3}$ will be a constant along adiabatic curves for this system.

![Pressure-Volume diagram](image)

a) Identify any steps during which no work is done.

b) How much heat flows in or out during steps "A" and "C"? (indicate the direction if your answer is non-zero).

c) How much heat flows in or out during steps "B" and "D"? (indicate the direction if your answer is non-zero). Hint: use the First law of Thermodynamics.

d) What is the ratio of total heat absorbed to total heat exhausted during one cycle?

e) What is the efficiency of this engine? (your answer should depend only on $V_1$ and $V_2$)
Problem #12

Consider a particle in thermal equilibrium with a reservoir at $T = 300K$. Suppose that there is one energy level at $E = -0.05$ eV, two energy levels at $E = 0$ eV, and one energy level at $E = +0.05$ eV.

a) What is the partition function for this particle?

b) What are the probabilities for the particle to be found with different energies?

c) What do you expect for an answer to part b for very high temperature, and what is the condition to call the temperature “very high.”

d) Explain why your answer in part-c makes physical sense.
Problem #13

The simplest form of graphene is a single layer of graphite, so it has a two-dimensional structure. Unlike most solids, the momentum-dependence of the electron energies in the conduction band are given by

\[ \varepsilon(\vec{p}) = v_0 p \]

and \( v_0 \) is the constant electron speed. For two dimensions,

\[ p = |\vec{p}| = \sqrt{p_x^2 + p_y^2} \]

a) Assume a single electron is placed in the conduction band and assume it is in thermal equilibrium at a temperature \( T \). Find the average energy of this electron. Your answer should be some constant time \( kT \)

It is more realistic to assume the graphene can have an arbitrary number of electrons in the conduction band which is determined by the Fermi distribution, \( f(\varepsilon) \). Assume the Fermi energy is zero.

b) Write the Fermi distribution for this case.

c) Assume a piece of graphite has an area \( A \). Then the internal energy of the electrons in the conduction band can be written as

\[ U = \frac{2A}{(2\pi\hbar)^2} \int \varepsilon(p)f(\varepsilon(p))d^2p \]

Find the internal energy. The answer should depend on \( A, \hbar, kT \) and \( v_0 \).

HINTS:

\[ \int_0^\infty x^n \exp(-x)dx = n! \]

\[ \int_0^\infty \frac{x^2}{\exp(x) + 1} dx \approx 1.8 \]
Problem #14

Suppose that you have a box of ideal gas at temperature $T$. The molecules in the box can exchange energy among themselves, but not with their surroundings.

**a.** Take the partition function for the system to be $Z$. Find the probability $P(E) \, dE$ that a molecule will have kinetic energy between $E$ and $E+dE$.

**b.** Show that the momentum distribution of the molecules is a Gaussian and also show that it can be factored into a product of three Gaussians.

**c.** Use the fact that a normalized one-dimensional Gaussian is written as $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ to find the partition function $Z$.

**d.** Find the probability distribution of the vector velocity. Hint: Do this transformation carefully. You may want to transform one component at a time.

**e.** What is the average velocity of the gas in any given direction? Does this make sense? Why or why not?
Problem #15

Consider a line of identical monovalent atoms. The total length of the atomic chain is \( L \) and the spacing between neighboring atoms is \( a \). We assume \( L >> a \).

(a) Sketch \( \varepsilon \) vs \( k \) for nearly-free electrons in a reduced zone scheme, including the first and the second energy bands.

(b) Describe the density of states \( D(\varepsilon) \) vs. \( \varepsilon \).

(c) Draw the dispersion curves \( \omega \) vs \( K \) of the acoustic phonons within the first Brillouin zone. Consider three polarizations (one longitudinal and two transverse).

(d) Determine the density of phonon mode \( D(\omega) \) vs. \( \omega \) for this one-dimensional chain of atoms.

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