Instruction:

Answer any three out of four questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Classical Physics

1. 
2. 
3. 
4. 

Student Identification Code: __________________________________

NOTE: The same code is to be used on all sections of the Comprehensive Examination taken in September 2006.
1) a) A short straight segment of wire of length $b$ carries a current $I$. The wire segment is centered at the origin and points in the $+x$ direction. Calculate the magnetic field produced by this wire segment as a function of $x$ and $y$ in the $x$-$y$ plane. Assume $\sqrt{x^2 + y^2} \gg b$.

b) Charge is uniformly distributed along a wire lying on the $x$-axis, with $-L \leq x \leq 0$. The total charge is $Q > 0$. Calculate the magnitude and direction of the electric field $\vec{E}(x_1)$ for points $x_1$ on the positive real axis. Show that your answer approaches the field of a point charge $Q$ at the origin when $x_1 \gg L$. 
Answer three out of four questions

2. At $t = 0$ a bullet is fired horizontally (direction of the $x$-axis) with a velocity $v_0$. The bullet is fired from the origin of an $x$-$y$ coordinate system. It is subject to gravity $(-g \hat{j})$ where $\hat{j}$ is the unit vector in the $y$-direction (up). It is also subject to a drag force $-b\vec{v}$ with $b > 0$.

a) Find the $x$ and $y$ components of the velocity as functions of time.
b) Find the $x$ and $y$ components of the position as functions of time.
c) Show that in the limit $b \to 0$, the $x$ and $y$ components of the position correspond to those of the bullet with no frictional force.
3. A satellite of mass $m$ is vertically launched to a height $h$ above the earth's surface. At that point, the satellite acquires an initial speed $v_0$ by firing a rocket horizontally, so that it orbits around the earth. Note that the orbit equation for the Kepler problem is

$$r = \frac{\alpha}{1 + \varepsilon \cos(\theta - \theta_0)}$$

where

$$\alpha = \frac{l^2}{mk} \quad \varepsilon = \sqrt{1 + \frac{2E_l^2}{mk^2}}$$

and $k = GmM = mgR^2$ where $R$ and $M$ are the radius and mass of the earth.

a) Express the angular momentum $l$ and the total energy $E$ of the satellite in terms of some of the following: $m$, $R$, $h$, $g$ and $v_0$.

b) Find the minimum initial speed $v_0$ so that the satellite will not fall into the earth.

c) Find the maximum $v_0$ for which the satellite can remain in orbit around the earth.
4. A piece of wire with a semicircular portion of radius $a$ lies on a $U$-shaped piece of wire. The two wires are in contact at points $P$ and $Q$, so current can flow around the loop enclosed by the two wires. The distance from the contact points to the left side of the loop (labeled $R$ and $O$) is $c$. The distance from the top to the bottom of the loop (from $R$ to $O$) is $b$. The loop is placed in a uniform magnetic field $B$ which is perpendicular to the plane of the loop. The wire PQ is rotated at an angular speed $\omega$ while keeping points P and Q fixed. At $t = 0$ the wire is in the position shown in the figure.

a) Express the total magnetic flux $\Phi$ encircled by the loop as a function of time when the $B$-field is constant.

b) Express the total magnetic flux $\Phi$ encircled by the loop as a function of time when the $B$-field varies with time as $B(t) = B_0 \cos(2\omega t)$

c) Find the electromotive force (emf) $\mathcal{E}(t)$ in the loop for case b).

d) Assume both pieces of wire have the same cross sectional area $A$, and conductivity $\sigma$. Assume there is negligible resistance at the points $P$ and $Q$ where the wires touch. Find the time dependant current in the wire for case b).
Classical Physics

Answer three of four questions

4. A piece of wire with a semicircular portion of radius $a$ lies on a U-shaped piece of wire. The two wires are in contact at points $P$ and $Q$, so current can flow around the loop enclosed by the two wires. The distance from the contact points to the left side of the loop (labeled $R$ and $O$) is $c$. The distance from the top to the bottom of the loop (from $R$ to $O$) is $b$. The loop is placed in a uniform magnetic field $B$ which is perpendicular to the plane of the loop. The wire PQ is rotated at an angular speed $\omega$ while keeping points $P$ and $Q$ fixed. At $t = 0$ the wire is in the position shown in the figure.

a) Express the total magnetic flux $\Phi$ encircled by the loop as a function of time when the $B$-field is constant.

b) Express the total magnetic flux $\Phi$ encircled by the loop as a function of time when the $B$-field varies with time as $B(t) = B_0 \cos(2\omega t)$

c) Find the electromotive force (emf) $E(t)$ in the loop for case b).

d) Assume both pieces of wire have the same cross sectional area $A$, and conductivity $\sigma$. Assume there is negligible resistance at the points $P$ and $Q$ where the wires touch. Find the time dependant current in the wire for case b).
Instruction:

Answer one from Questions 5 and 6
and one from Questions 7 and 8

Please check below the two problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the two books at the end of the exam.

Advanced Classical Physics

5. __________

6. __________

7. __________

8. __________

Student Identification Code: ________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in September 2006.
Answer one from Questions 5 and 6 and one from Questions 7 and 8

5. A uniform solid ball of mass $m$ and radius $a$ rolls without sliding on an identical ball whose center is fixed in space. Initially, one ball is directly above the other. The bottom ball is free to rotate. The top ball is subjected to gravity, and rolls off the bottom ball. The system is initially nearly at rest, except for a very small initial motion of the top ball.

a) Calculate the moment of inertia for a uniform solid ball of mass $m$ and radius $a$ about the center of mass to verify that $I = \alpha ma^2$. What is $\alpha$?

The center of the top ball moves on the sphere of radius $2a$ with angular speed $\dot{\theta}$. The top ball and the bottom ball rotate with angular speeds $\psi$ and $\phi$, respectively.

b) Express $\dot{\theta}$ in terms $\psi$ and $\phi$.

c) Obtain and simplify the equations of motion, to obtain

\[
\frac{d^2\theta}{dt^2} = K \sin(\theta)
\]

and find $K$.

d) Multiply the equation of part c) by $\frac{d\theta}{dt}$ and integrate over time to find the velocity of the top ball as a function of $\theta$.

e) At what $\theta$ will the top ball separate from the bottom ball?

NOTE: One can do this problem using Lagrange multipliers, but it is not necessary.
Answer one from Questions 5 and 6 and one from Questions 7 and 8

6. The diagram shows a vertical post which can rotate about its vertical axis with angular speed $\omega$. There is a rod of length $L$ connected to the post such that it must rotate with the post, but is free to swing away from the post. A ball of mass $M$ is attached to the end of the rod. Neglect friction in all parts of this problem.

a) If the rod is massless, what is the angle $\theta$ as a function of the angular speed $\omega$ of the post? HINT: Write a force diagram for the mass that includes the tension on the ball from the rod, and the gravitational force on the ball.

b) If instead of being massless, assume the rod has mass $m$ uniformly distributed along its length. What is the moment of inertia of the rod-ball system relative to the axis along the post?

c) When the rod has a mass $m$, what is the angle $\theta$ as a function of the angular speed $\omega$ of the post?

d) Find the total energy of the system as a function of $\omega$ for the case that the rod has mass $m$. HINT: Remember that the angle $\theta$ will be different for the two different rotation rates.
Answer one from Questions 5 and 6 and one from Questions 7 and 8

7. An electromagnetic wave of amplitude $A$ and intensity $I_0 = |A^2|$ propagates along the $z$-axis. Its electric field is polarized along the $x$-axis. The wave encounters a polarizer (denoted “F” for first) which transmits only the electric field component parallel to the polarizer axis (the optic axis). Its optic axis is at a 45 degree angle from the $x$-axis toward the $y$-axis.

a) Find the ratio of the transmitted intensity $I$ to the incident intensity $I_0$. What is the direction of polarization of the transmitted beam?

The wave continues on to a another polarizer (denoted “L” for last) whose optic axis is rotated at a 135 degree angle from the $x$-axis toward (and past) the $y$-axis.

b) Find the ratio of the transmitted intensity $I$ to the incident intensity $I_0$ for the light which passes through both polarizers.

A third polarizer (denoted “M” for middle) is placed between the first and second polarizer. Its optic axis is at 90 degrees, parallel to the $y$-axis. The beam passes through all three polarizers.

c) Find the ratio of the transmitted intensity $I$ to the incident intensity $I_0$.

d) The middle polarizer is allowed to rotate, so its optic axis is at an angle $\theta$ with respect to the $x$-axis. Find the intensity of the light which passes through all three polarizers as a function of the middle polarizer angle $\theta$. Sketch your result as a function of $\theta$ for $0 \leq \theta \leq 2\pi$. 

---

---
Answer one from Questions 5 and 6 and one from Questions 7 and 8

8. PART I
A positive charge \( Q \) is placed on the z-axis at \( z = +P \) outside a grounded conducting sphere of radius \( a \) centered at the origin. The electrostatic potential outside the sphere can be obtained as the sum of the potential produced by \( Q \) plus the potential produced by an "image charge" \( q \) at \( z = b \). The magnitude and position of the image charge can be obtained by demanding that the electrostatic potential vanish on the surface of the sphere.

a) Find \( q \) in terms of \( Q, R \) and \( a \).
b) Find \( b \) in terms of \( P \) and \( a \).

PART II
A charge \( q \) is placed at \( z = b \) and a charge \( -q \) is placed at \( z = -b \). Let \( r \) be the distance of a point from the origin, and let \( \theta \) be the angle of that point away from the z-axis.

c) Find electric potential \( \psi(r, \theta) \) to lowest order in \( 1/r \) when \( r \gg b \).

PART III
The electrostatic potential of a grounded sphere in a uniform electric field \( E \) directed along the negative z-axis can be obtained from the results of PART I and PART II. Place a charge \( Q \) at \( z = +R \) and a charge \( -Q \) at \( z = -R \). Take the limits
\[
Q \to \infty; \quad R \to \infty; \quad \frac{2Q}{4\pi\varepsilon_0 \cdot R^2} = |E|
\]
d) Find the electric potential which is the sum of the potentials due to \( +Q, -Q \), and the image charges \( +q \) and \( -q \).
e) Find the electric field at the surface of the sphere.
f) Use this field to find the charge density on the surface of the sphere.
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part III

Thursday, September 7, 2006
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Statistical and Thermal Physics

9. __________
10. __________
11. __________
12. __________

Student Identification Code: ________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in September 2006.
Answer three of four questions

9. Two perfect monatomic gases with the same pressure $P$ and the same number of particles $N$, but with different temperatures $T_1$ and $T_2$ are confined in two vessels of volume $V_1$ and $V_2$. The vessels are thermally insulated from the outside world. The two vessels are connected so particles and energy can flow between them. Find the change in the total entropy after the system has reached thermal equilibrium for Case a) and Case b).

Case a) The two gases are made of identical atoms.

Case b) The atoms in the two gases are distinguishable.
Answer three of four questions

10. a) Write the Fermi function which gives the thermal equilibrium probability that a single-electron state with energy \( \varepsilon \) will be occupied. Your answer will depend on the energy \( \varepsilon \), chemical potential \( \mu \) and the Kelvin temperature \( T \). Alternatively, you can use \( \beta = 1/(kT) \) where \( k \) is Boltzmann's constant.

b) A large number of electrons \( (N) \) are placed in a box of volume \( V \). Ignore interactions between the electrons, so the energy of each electron is just its (non-relativistic) kinetic energy. The internal energy \( U \) of these electrons in thermal equilibrium is the sum of single-electron energies. Using the Fermi function and a sum over momentum and spin states, this internal energy can be expanded in a power series in the temperature, \( T \).

\[
U = A + BT + CT^2 + \ldots
\]

The coefficients \( A \), \( B \) and \( C \) depend on \( N \) and \( V \), but they are not functions of the temperature.

b) Find the coefficients \( A \) and \( B \).

c) Give an estimate of the coefficient \( C \).
Answer three of four questions

11. The Earth's greenhouse gases absorb radiation from the earth's surface, preventing it from escaping into space. This radiation is emitted back to the surface and keeps the surface at a temperature 35K higher than it would be if it were a blackbody emitter.

A Hurricane is a heat engine that takes warm moist air from the sea surface and lifts it into the upper atmosphere where it can lose energy by radiating into space. The figure below (K. Emanuel, Physics Today, Aug. 2006) shows what happens. Warm sea air at 300K is drawn from the edge of the hurricane, A, toward the low pressure eye of the storm, B. As this happens, the air is approximately in contact with the sea, which is a heat reservoir. So the gas expands isothermally. It then is lifted up along the storm's eyewall from B to C and expands adiabatically. In the upper atmosphere, it radiates energy into space and cools to 200K and is compressed approximately isothermally, represented symbolically as C to D. The air is then moves down from D to A and during this process is compressed adiabatically.

![Diagram of Hurricane Cycle](image)

- a) Assume that at point A, the surface air pressure is normal at 101.3 kPa, and that at point B, the air pressure is 92 kPa as measured in the eye of Hurricane Katrina. What is the volume of 1 liter of air at point A (assume ideal gas) after it has been transported to the eye of the hurricane at point B?

- b) Assume the pressure at point C is again normal at 12.1 kPa at 15 km above the surface. What is the volume of the gas at point C?

- c) Make a diagram of the cycle with axes of Pressure versus Volume for 1 liter of air. Be sure to label all four points A, B, C, D and the four processes, such as adiabatic expansion, etc.

- d) This is approximately a Carnot cycle and the Hurricane is a heat engine. Compute the Carnot efficiency of a hurricane.
12. A tiny mirror suspended by a wire forms a torsional pendulum with a torsion constant

\[ k = 2.47 \times 10^{-10} \text{ kg (meter)}^2 \sec^{-2}. \]

The wire-mirror system is placed in a sealed vessel which is held at a temperature of 300K. A beam of light is reflected off the mirror and onto a wall 0.05 meters from the face of the mirror. A top view of the mirror and reflected light-beam is shown in the figure.

a) How much potential energy does the pendulum gain if the mirror twists by an angle \( \theta \)?

b) Use the equipartition theorem to evaluate the mean-square value of the angular twist due to thermal fluctuations, \( \langle \theta^2 \rangle \).

c) What is the root-mean-square amplitude of the light-spot’s motion on the wall?

d) Assume the light can be focused to spot roughly equal to its wavelength, \( \Delta x = 5 \times 10^{-7} \text{ meters} \). At approximately what temperature would the thermal motion be observable with this system?

Boltzmann’s constant is \( k_B = 1.38 \times 10^{-23} \text{ Joules/degree Kelvin} \).
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part IV

Thursday, September 7, 2006
1:00 - 3:00 PM

Instruction:

Answer any three out of seven questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Modern Physics

13. __________
14. __________
15. __________
16. __________
17. __________
18. __________

Student Identification Code: ________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in September 2006.
Modern Physics

Do three out of six problems

13 a) A gold ($^{197}$Au) target is 20 nm thick

What is the thickness of the target in units of ($\text{gold atoms/cm}^2$)?

NOTE: The density of gold is 19.3 grams/cm$^3$ and Avogadro's number is $6.02 \times 10^{23}$ mol$^{-1}$

The gold target of a) is used in a Rutherford backscattering experiment at a scattering angle of 180 degrees. It is bombarded with a 2MeV He$^+$ beam.

b) What is the energy of the He ion after scattering?

c) What is the energy of the gold atom after scattering?

NOTE: At 180 degrees, the kinematic factor is $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$.

The total charge delivered to the target by the He$^+$ beam is 5 microcoulombs.

d) How many He$^+$ ions have bombarded the target?

NOTE: The electron charge is $e = 1.6 \times 10^{-19}$ Coulombs

The differential cross section for Rutherford scattering at 180 degrees is 

$$\sigma(\theta) = \frac{d\sigma}{d\Omega} = 32 \times 10^{-24} \text{cm}^2 / \text{sr}$$

A Rutherford backscattering detector subtends an angle of 2 msr ($2 \times 10^{-3}$ sr).

e) What is the total number of He$^+$ ions backscattered from the gold into the detector?

f) What is the approximate standard deviation to the answer of part d). Assume the experiment delivering 5 microcoulombs is done only once.
Do three out of six problems

14. Briefly describe the physics behind the following observed phenomena:
   a) A reflective high-energy electron diffraction pattern (RHEED) from a smooth crystal surface consists of streaks instead of diffraction spots.
   b) X-ray diffraction cannot detect magnetic ordering, but neutron diffraction can.
   c) A direct correlation is found between the diffusion constant and the electrical conductivity in alkali halides.
   d) Heating a metal bar produces an expansion in length ($\Delta l/l$) which is slightly larger than the change in the lattice constant ($\Delta a/a$).
   e) All noble gases and many diatomic gases, such as $\text{H}_2$, are diamagnetic.
   f) The magnetization of Fe by an external applied magnetic field shows a hysteresis loop.
15. Two photons each have energy $E$. They collide at an angle $\theta$ and create a particle of mass $M$. Express $M$ in terms of $E$ and $\theta$. 

\[ M \]
Do three out of six problems

16. In 1899, Becquerel discovered that some heavy mass nuclei were unstable and decayed.
a) How did he determine the nature of the rays emitted in this decay?
b) Free neutrons are unstable. Write the decay process.
c) The electrons emitted in the beta decay of some nuclei have a continuous energy spectrum. This appeared to violate conservation of energy. How was this problem resolved?
d) Some beta decays are allowed and some are forbidden. What properties of the initial and final states determine whether or not the decay is forbidden.
e) The figure below show the gamma decay of a nucleus. Calculate the energies and the nature of the gamma-ray multi-polarities (E and M) of the photon decays numbered 1, 2, 3 and 4.

<table>
<thead>
<tr>
<th>J</th>
<th>π</th>
<th>MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>4+</td>
<td></td>
<td>3.50</td>
</tr>
<tr>
<td>3-</td>
<td></td>
<td>2.00</td>
</tr>
<tr>
<td>2+</td>
<td></td>
<td>0.85</td>
</tr>
<tr>
<td>0+</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
17. a) List the ground-state configurations for the following atoms:

Li (Z = 3)
B (Z = 5)
N (Z = 7)
Na (Z = 11)
K (Z = 19)

b) List the L, S, and J quantum numbers for these atoms.

c) What transition corresponds to the lowest frequency line in the absorption spectrum of Na?

d) This frequency line is a doublet. What mechanism splits the energy levels?
Magnetotactic bacteria possess small organelles called magnetosomes that contain on the order of 20 magnetite crystals. These crystals serve to orient bacteria in the Earth’s magnetic field. Each magnetite crystal has a magnetic moment of about $6.40 \times 10^{17} \text{ A m}^2$. The Earth’s magnetic field is $5 \times 10^{-5} \text{T}$.

a) Compute the total magnetic moment of a magnetosome assuming that the magnetic moments of the crystals are all aligned.

b) Write the expression for the orientation energy of the bacterium as a function of its orientation angle $\theta$ with respect to the Earth’s magnetic field.

c) Compute the ratio of the orientation energy with respect to $kT$.
Assume $T = 300K$. Boltzmann’s constant is $k = 1.38 \times 10^{-23} \text{ Joule/Kelvin}$

d) This bacterium has a flagellum that helps it swim with speed $v_0$. In the absence of a magnetic field, the bacterium will swim in any direction with equal probability. The probability that it will be moving in the direction described by a solid angle of dimension $d\Omega = \sin\theta \, d\theta \, d\phi$ centered at $(\theta, \phi)$ is $p(\theta, \phi) = d\Omega/4\pi$. What is the probability that it is moving at an angle $\theta$ irrespective of $\phi$?
In other words, given $p(\theta, \phi)$, what is $p(\theta)$?

e) This probability $p(\theta)$ is altered by the torque induced by the Earth’s field. Using the orientation energy and its relation to $kT$, write the probability of finding the bacterium moving at an angle $\theta$ with respect to the Earth’s field.

f) Find the average velocity of the bacterium in the direction of the Earth’s field.
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part V

Saturday, September 9, 2006
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Quantum Mechanics

19. 

20. 

21. 

22. 

Student Identification Code: ________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in September 2006.
Answer three out of four questions

19. A rigid rotor (with a non-zero magnetic moment) is immersed in a uniform magnetic field in the z-direction. The Hamiltonian for the system is

\[ \hat{H} = \frac{1}{2I} \hat{J}^2 + \omega \hat{L}_z \]

Initially (at \( t = 0 \)), the wave function which describes the orientation of this rotor is

\[ \psi(\theta, \phi; 0) = \frac{3}{4\pi} \sin \theta \sin \phi \]

a) Find the wave function \( \psi(\theta, \phi; t) \) which describes this rotor at later times.
b) Find the expectation value of the angular momentum along the x-axis, \( \langle \hat{L}_x \rangle \) for the system described by the time-dependent wave function of part a).

NOTES: The spherical harmonics are the eigenfunctions of the squared angular momentum operator and the z-component of the angular momentum operator.

\[ \hat{L}_z Y_{l,m}(\theta, \phi) = \hbar m Y_{l,m}(\theta, \phi) \]

The spherical harmonics for \( l = 1, 2 \) are

\[ Y_{0,0} = \sqrt{\frac{1}{4\pi}} \]
\[ Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta \]
\[ Y_{0,1} = \frac{i}{\sqrt{8\pi}} \sin \theta \exp(i\phi) \]

Also,

\[ \hat{L}_x = \frac{1}{2} (\hat{L}_+ + \hat{L}_-) \]

and

\[ \hat{L}_z Y_{l,m} = \hbar \sqrt{l(l+1) - m(m \pm 1)} \ Y_{l,m\pm1} \]
20. A particle with mass $m$ and energy $E$ encounters a potential barrier with energy $V > E$ and width $a$.

a) What are the conditions that must be met at the two boundaries ($x = 0$ and $x = a$)?

b) What property does the wave function have for $x > a$ if the incident particle is incident from the left?

c) Find the probability $P$ that the particle will tunnel through the barrier.

The answer to part c) can be simplified when

$$\Gamma a = \sqrt{\frac{2m}{\hbar^2}} (V - E) a \gg 1$$

With the simplification, the tunneling probability becomes

$$P \equiv K \exp(-C \cdot \Gamma a)$$

d) What is the constant $C$ in the above expression?
Quantum Mechanics

Fall 2006

Answer three out of four questions

21. A particle with mass $m$ and charge $q$ moves along the $x$-axis. It is in a harmonic oscillator potential $V(x)$. The minimum of $V(x)$ is at $x = 0$, and $V(0) = 0$. The angular frequency of this harmonic oscillator system is $\omega$.

a) Write the Hamiltonian $H$ for this system.

b) What are the energy levels $\epsilon_n$ of this system?

An electric field $E$ (directed along the $+x$ axis) is applied to this system. This adds a perturbation $H'$ to the Hamiltonian.

c) What is $H'$?

d) The ground state wave function $\psi(x)$ of the harmonic oscillator is symmetric about the minimum in the potential. Use this fact to find the first-order perturbation correction to the ground state energy produced by $H'$.

The Hamiltonian $(H + H')$ also describes a harmonic oscillator.

e) Find the energy levels of the system described by $(H + H'), \epsilon_n(E)$, as functions of the electric field $E$.

The normalized ground state of the original system, $\psi(x)$, and the ground state of the perturbed system $\phi(x)$ are related by

$$\phi(x) = \psi(x + a)$$

f) Find $a$.

g) If the system is initially in the ground state, and the perturbation is applied suddenly, obtain a formula for the probability that the particle remains in the ground state. Express this result as an integral which depends on $\psi(x)$. 
22. A particle of mass $m$ moves in one dimension, subject to the potential

$$V(x) = \begin{cases} 
0 & 0 \leq x \leq L \\
\infty & x < 0 \text{ or } x > L 
\end{cases}$$

a) Starting with the Schrödinger equation, showing each step clearly, derive the energies $\varepsilon_n$ and energy eigenfunctions $\psi(x)_n$ for this system.

b) Derive the expectation value of $x$ for each $n$. That is, find

$$\langle x \rangle_n = \int_0^L \psi(x)_n^* x \psi(x)_n \, dx$$

c) Derive the expectation value of $x^2$ for each $n$. That is, find

$$\langle x^2 \rangle_n = \int_0^L \psi(x)_n^* x^2 \psi(x)_n \, dx$$

d) Show that for large $n$,

$$\langle x^2 \rangle_n - \langle \langle x \rangle_n \rangle^2 \to \frac{L^2}{12}$$

NOTE:

$$\int y \sin^2(y) \, dy = \frac{1}{8} \left( 2y^2 - 2y\sin(2y) - \cos(2y) \right)$$

$$\int y^2 \sin^2(y) \, dy = \frac{1}{24} \left( 4y^3 - 6y^2 \sin(2y) + 4\sin(2y) - 6y\cos(2y) \right)$$
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part VI

Saturday, September 9, 2006
1:00 - 3:00 PM

Instruction:

**Answer any two out of three questions**

Please check below the two problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the two books at the end of the exam.

Advanced Quantum Mechanics

23. 

24. 

25. 

Student Identification Code: _______________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in September 2006.
23. a) An electron (spin-$1/2$ fermion with mass $m$) moves on a line of length $L$ with periodic boundary conditions. So 

$$-L/2 \leq x \leq L/2$$

with 

$$\psi(-L/2) = \psi(+L/2)$$

$$\left. \frac{d\psi}{dx} \right|_{x \rightarrow -(L/2)^+} = \left. \frac{d\psi}{dx} \right|_{x \rightarrow +(L/2)^-}$$

The potential on the ring is zero.

PART I, basics

a) Find the lowest energy state on this ring which is symmetric, so 

$$\psi(x) = \psi(-x)$$

b) Find the lowest energy state on this ring which is anti-symmetric, so 

$$\psi(x) = -\psi(-x)$$

PART II, non-zero potential

The potential on the ring is changed. A localized delta-function potential is added at the origin, so 

$$V(x) = A\delta(x)$$

Assume $A > 0$.

c) What are the units of $A$?

d) Find the lowest energy state on this ring which is symmetric. 

HINT: Try 

$$\psi(x) = \cos(k|x| - \phi)$$

Determine the phase $\phi$ using the Schrödinger equation with the potential $V(x)$. Then pick the smallest $|k|$ which satisfies the boundary conditions at $x = \pm L/2$.

e) For sufficiently small $A$, the phase will be small, and $\tan \phi \rightarrow \phi$. Find the energy of this lowest energy symmetric state using the approximation $\tan \phi \rightarrow \phi$.

PART III, non-interacting electrons

The delta-function potential at the origin is removed. Two electrons are placed on the ring.

f) What is the lowest total energy eigenstate of this two electron system when both electrons have “spin-up”? 

NOTE: Remember the Pauli principle. The energy is the sum of the two single-electron energies.
g) What is the lowest energy eigenstate when the system is in a “spin singlet” state (antiparallel spins)?

PART IV, interacting electrons
Assume the electrons interact through a delta-function potential

\[ V(x_1, x_2) = A \delta(x_1 - x_2) \]

With \( A > 0 \). Assume the potential is sufficiently small, so \( \tan \phi \to \phi \).

h) What is lowest total energy of this system when the spins are parallel?

i) What is the lowest energy when the system is in a spin-singlet state?

NOTE: You can use center of mass and relative separation coordinates. The mass associated with the motion of the center of mass is \( 2m \). The mass associated with the relative separation is the reduced mass, \( m/2 \).
24. A hydrogen atom in the ground state is placed in a uniform time-dependent electric field in the z-direction. The field is turned on at \( t = 0 \). Thus an electron in this field is subject to the perturbation

\[
\hat{H}'(t) = \begin{cases} 
E_0 \exp\left(-\frac{t}{\tau}\right) r \cos(\theta) & t > 0 \\
0 & t \leq 0
\end{cases}
\]

with \( \tau > 0 \).

a) Calculate the atomic matrix elements

\[
H'(t)_{f,i} = \langle 2,l,m | \hat{H}'(t) | 1,0,0 \rangle
\]

The probability amplitude of a transition from the \( i \)-th state (initial) to the \( f \)-th state (final) due to this time-dependent perturbation is

\[
c_f(t) = \int_{-\infty}^{t} \exp\left(\frac{\mathcal{E}_f - \mathcal{E}_i}{\hbar} t'\right) \hat{H}'(t')_{f,i} dt'
\]

where \( \mathcal{E}_f \) and \( \mathcal{E}_i \) are the energies of the final and initial states.

b) What is the probability that the atom undergoes a transition to the \( 2s \) state as \( t \to \infty \)?

c) What is the probability that the atom undergoes a transition to the \( 2p \) state as \( t \to \infty \)?

NOTE:

\[
\psi_{n\ell m} = R_n(r)Y_l^m(\theta, \phi)
\]

\[
\int_0^\pi d\theta \int_0^{2\pi} d\phi Y_l^m(\theta, \phi) \sin \theta d\phi d\theta = \delta_{\ell,0} \delta_{m,0}.
\]

\[
Y_0^0 = \frac{1}{\sqrt{4\pi}}
\]

\[
Y_1^0 = \frac{3}{\sqrt{4\pi}} \cos \theta
\]

\[
\int_0^\infty \exp\left(-\frac{x}{a}\right) x^n dx = n! a^{n+1}
\]

For the hydrogen atom with Bohr radius \( a \),

\[
R_{10} = 2a^{-3/2} \exp\left(-\frac{r}{a}\right)
\]

\[
R_{21} = \frac{1}{2\sqrt{6}} a^{-5/2} r \exp\left(-\frac{r}{2a}\right)
\]

\[
\mathcal{E}_n = -\frac{\hbar}{2m_e a^2} \frac{1}{n^2}
\]

where \( m_e \) is the electron mass and \( n \) is any positive integer.
Answer two out of three questions

25. A beam of particles of mass \( m \) is scattered by a potential

\[
V(r) = \begin{cases} 
V_0 & r \leq a \\
0 & r > a 
\end{cases}
\]

The Born approximation for the scattering amplitude is

\[
f(\theta) = -\frac{2m}{\hbar^2 K} \int_0^r V(r) \sin(Kr)rdr
\]

where \( K = 2k \sin\left(\frac{\theta}{2}\right) \) and \( k \) is the incident wave number. The incident energy is

\[
E = \frac{\hbar^2 k^2}{2m}.
\]

a) Using the Born approximation, calculate the scattering amplitude.

b) Obtain the differential scattering cross section, \( \sigma(\theta) = \frac{d\sigma}{d\Omega} \).

c) Find an approximation for this differential scattering cross section for low energies where terms containing \((ka)^4\) and higher powers may be ignored.

d) Is the scattering isotropic (angle-independent) for all energies? Explain briefly.

e) Integrate the differential scattering over solid angles to obtain the total scattering cross section, \( \sigma_t \) for the low energy case of part c).