INSTRUCTIONS

Answer any three out of four questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Classical Physics

1. ____________
2. ____________
3. ____________
4. ____________

Student Identification Code: ___________________________________________________________________

NOTE: The same code is to be used on all sections of the Comprehensive Examination taken in January 2004.
Do three out of four problems

1) A ring of radius $R$ and mass $M$ carries a charge $Q$. It rotates about its axis at a constant angular frequency $\omega$. A uniform magnetic field $B$ is established with its direction making an angle $\theta$ with respect to the rotation axis of the ring. Determine the precession rate and precession period of this ring.

NOTE:
The magnetic moment of a current loop $|\vec{\mu}|$ is the product of the current times the area.
The torque on a magnetic moment in a magnetic field $\vec{B}$ is $\vec{N} = \vec{\mu} \times \vec{B}$. The time derivative of the angular momentum of an object is given by the torque on that object.
Classical Physics

Spring 2004

Answer three out of four problems

2) A bullet of mass $m$ is shot up vertically with a velocity $v_0$. This bullet is subject to a drag force (due to friction with the air) given by

$$F = -cv^2$$

where the minus sign means the drag force is the opposite direction to the velocity, $v$.

a) Calculate (exactly) the velocity of this bullet as function of height, $x$ as it rises to its maximum height.

HINT:

$$a = \frac{dv}{dt} = \left( \frac{dv}{dx} \right) \left( \frac{dx}{dt} \right) = v \frac{dv}{dx} = \frac{1}{2} \frac{d}{dx} v^2$$

b) Calculate the maximum height the bullet goes before falling back to earth.

c) If there were no drag force, you can calculate the maximum height using conservation of energy. Show that in the limit

$$c \to 0$$

the maximum height of part b) becomes the height obtained from conservation of energy.
Answer three out of four problems

3) In each of the four following situations, you are asked to obtain a field. You must justify your answer. You will be given no credit for just writing a correct final answer. NOTE: Fields are vectors. Be sure to specify both magnitude and direction.

a) A charge $+Q$ is placed at the origin. What is $\vec{E}$ at points $P_1$ and $P_2$ which are a distance $l$ from the origin along the $+x$ and $+y$ axes?

\[ \text{Diagram:} \]

b) A current element $d\vec{s}$ at the origin carries a current $I$ along the $+x$ axis. What is $\vec{B}$ at points $P_1$ and $P_2$?

\[ \text{Diagram:} \]
c) An infinite line on the $x$-axis carries a charge/length $\lambda$. What is $\vec{E}$ at a point $P$ a distance $l$ from the line in the $+y$ direction?

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\[ \begin{array}{c}
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\[ \begin{array}{c}
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\[ \begin{array}{c}
\text{I}
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d) An infinite line on the $x$-axis carries a current $I$. What is $\vec{B}$ at a point $P$ a distance $l$ from the line in the $+y$ direction?

\[ \begin{array}{c}
P \\
|
\quad l \\
\hline
\text{X}
\end{array} \]

\[ \begin{array}{c}
\text{I}
\end{array} \]
Answer three out of four problems

4) A bug of mass $m$ slowly climbs up on the inner wall of a cylinder of radius $a$. The bug starts from the bottom of the cylinder. Assume that the friction coefficient between the bug and the wall is $\mu$.

a) Find the maximum height $h$ that the bug can reach.

The cylinder is rotated about its axis with an angular frequency $\omega$. This carries the bug up the wall a height $h' > h$ before it starts to slip.

b) Find the value of $\omega$ for which slipping starts when $h = a$. 

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[Diagram of a cylinder with radius $a$ and a bug climbing up with height $h$.]
DEPARTMENT OF PHYSICS

University at Albany
State University of New York

Comprehensive Field Examination

Part II

Tuesday, January 20, 2004
1:00 - 3:00 PM

Instruction:

Answer one from Questions 5 and 6
and one from Questions 7 and 8

Please check below the two problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the two books at the end of the exam.

Advanced Classical Physics

5. ___________

6. ___________

7. ___________

8. ___________

Student Identification Code: ____________________________________________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in January 2004.
Answer one of questions 5 and 6, and one of questions 7 and 8.

5) A hollow conducting sphere of radius \( a \) is centered at the origin. A point charge \( q \) is located a distance \( d > a \) from the center of the sphere.

For parts a), b) and c), assume the conducting sphere is grounded.

a) What is the electrostatic potential \( \Phi(\vec{r}) \) inside the sphere?

b) Obtain an expression for the electrostatic potential \( \Phi(\vec{r}) \) at an arbitrary point outside the sphere.

c) Calculate the induced surface charge density \( \sigma \) on the sphere.

d) Find the force on the charge \( q \) due to the grounded conducting sphere.

The conducting sphere is detached from ground. The total charge on the sphere (including any induced charge due to \( q \)) is now \( Q \). The new \( \Phi(\vec{r}) \) will depend on \( q \), \( Q \) and \( d \).

e) Find the new \( \Phi(\vec{r}) \) inside the sphere.

f) Find the new \( \Phi(\vec{r}) \) at an arbitrary point outside the sphere.
Answer one of questions 5 and 6, and one of questions 7 and 8.

6) The electric field of a plane electromagnetic wave in vacuum is
\[ \vec{E}(\vec{r}, t) = E_0 \hat{x} \Re \{ \exp(i(kz - \omega t)) \} \]
where \( E_0 \) is a positive real amplitude and \( \hat{x} \) is a unit vector along the x-axis.

a) Use Maxwell’s equations to find a relation between \( k \) and \( \omega \). (Assume they are both positive.)

b) Use Maxwell’s equations to find the magnetic field of this wave.

A free electron (charge \(-e\) and mass \(m\)) is in the path of this electromagnetic field. Assume that at \( t = 0 \) the electron is at rest at the origin. The electromagnetic field causes the electron to move.

c) Considering only the influence of the electric field, find the velocity of the electron for \( t > 0 \).

d) When the electromagnetic wave intensity is small, one can use the electron velocity of part c), to obtain the approximate additional (time dependent) velocity given to the electron by the magnetic field. Find this additional velocity, and specify its direction.

NOTE: Assume the wave length of the electromagnetic field is large compared to the distance traveled by the electron.
Answer one of questions 5 and 6, and one of questions 7 and 8.

7) A point-mass slides back and forth (under the influence of gravity) on the top surface of a block. The block slides without friction on the top of a table. The point-mass and the block have the same mass, \( M \). The top surface of the block is described by the equation

\[
y = y_0 + \alpha x^2
\]

where \( x \) is the horizontal distance from the center of the block. Consider motion only in the \( x \)-direction.

a) Write the Lagrange equations for the motion of the point mass and the block. You do not need to solve these equations; just set them up.

b) Find the frequency of the small amplitude oscillation of this system.

NOTE: It is not required that you start with the Lagrange equations of part a) to find the small amplitude oscillations of part b). Other methods may be easier.
Answer one of questions 5 and 6, and one of questions 7 and 8.

8) A particle of mass $m$ moves in a circular orbit of radius $R$ under the influence of an attractive central force $F(r)$ directed toward a point $O$ on the circle, as shown below. The particle’s angular momentum (measured from the origin $O$) is $L$.

a) Find the form of the potential $V(r)$ by using the equation for the orbit. Assume the potential vanishes as $r \to \infty$. Your answer should depend on $r$, $R$, $m$ and $L$.

b) Find the total energy.

c) What is the time needed for the particle to move from the point $B$ on the opposite side of the circle to the point $O$?

NOTE: Even though the potential is singular at $O$ assume non-relativistic mechanics are valid.
DEPARTMENT OF PHYSICS

University at Albany
State University of New York

Comprehensive Field Examination

Part III

Thursday, January 22, 2004
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Statistical and Thermal Physics

9. __________
10. __________
11. __________
12. __________

Student Identification Code: ____________________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in January 2004.
9)
a) A Carnot engine works between temperatures $T_1$ and $T_2$. It drives a Carnot refrigerator that works between temperatures $T_3$ and $T_4$. What is the ratio of the heats $Q_3/Q_1$? (See figure below) Express your answer in terms of the temperatures.
b) An Otto cycle consists of adiabatic compression and expansion (steps $A \rightarrow B$ and $C \rightarrow D$) and heating and cooling at constant volume (steps $B \rightarrow C$ and $D \rightarrow A$). Its efficiency is defined as the net work obtained from one cycle divided by the heat added in step $B \rightarrow C$.

i) Express the efficiency in terms of the internal energy at points $A$, $B$, $C$, $D$.

Assume the cycle is performed on an ideal gas, so the internal energy is proportional to the temperature, and for an adiabatic process,

$$PV' = \text{constant}$$

ii) Obtain an expression for the efficiency for the ideal gas Otto cycle which depends on only two temperatures.
Do three out of four problems

10) In sixteenth century Germany, the "rood" was defined as the distance obtained by placing the left foot of 16 random men in a row. Assume these men had left foot sizes varying randomly and uniformly between 11 inches and 13 inches. That means the distribution of foot sizes is as shown below.

\[ P(x) \]

To test this unit of length, one could assemble a large number of groups of 16 German men and find

\[ \sigma^2 (rood) = \langle (1\text{rood} - 16\text{feet})^2 \rangle \]

where the \( \langle \cdots \rangle \) means a statistical average over all the groups, each rood is a experimental measure, but 16 feet is the exact distance.

a) Obtain an approximate value for \( \sigma (rood) \). Justify any approximations you might make.

b) Knowing the length of a rood also gives an experimental determination of the foot, with a statistical error \( \sigma (foot) \). What is \( \sigma (foot) \)?
Do three out of four problems

11) a) Find the kinetic energy (in eV) of a hydrogen molecule near the surface of the earth with a speed equal to the escape velocity \( v_0 \equiv 11,000 \text{m/s} \).

NOTE:
For a proton,

\[ m_p c^2 \equiv 931 \text{MeV} \]

where \( c = 3 \times 10^8 \text{m/s} \).

b) If the temperature of the atmosphere is taken to be room temperature, so \( kT = \frac{1}{40} \text{eV} \), what is the ratio

\[ r_o = \frac{\text{escape energy}}{kT} \]

c) Assume a Boltzmann distribution of velocities with a temperature \( kT = \frac{1}{40} \text{eV} \). What is the approximate probability that a hydrogen molecule will be moving fast enough to escape the earth’s gravity?

NOTE:
For large \( x_0 \)

\[ \int_{x_0}^{\infty} \exp(-x^2) x^n dx \rightarrow \frac{1}{2} x_0^{n-1} \exp(-x_0^2) \]

d) Give a rough estimate of the probability that an oxygen molecule would have enough energy to escape the earth’s gravity.

e) Do these results explain why there is so little hydrogen on the surface of the earth, and so much hydrogen in the sun?
Do three out of four problems

12) A system of $N$ non-interacting particles, each with spin 1/2 and a magnetic moment $\mu$ is placed in a magnetic field $B$. The spins can be either "up" (in the direction of $B$) or "down" (opposing the direction of $B$). The corresponding energies of the "up" and "down" spins are $-\mu B$ and $+\mu B$.

a) For both the low-temperature $\left(\frac{kT}{\mu B} \rightarrow 0\right)$ and high-temperature $\left(\frac{kT}{\mu B} \rightarrow \infty\right)$ limits, determine the energy and entropy of this $N$-spin system.

b) The system is initially at $T = 4K$ in a $B$-field of 1Tesla. The system is isolated (i.e. adiabatic process) and the magnetic field is slowly reduced to 0.01Tesla. What will be the final temperature?

c) The graph below describes the entropy as a function of the internal energy of this system. Explain why this system can have a negative temperature. If this system with negative temperature is in thermal contact with a thermal reservoir with positive temperature, will heat flow into the system or out of the system?
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part IV

Thursday, January 22, 2004
1:00 - 3:00 PM

Instruction:

Answer any three out of seven questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Modern Physics

13. ___________
14. ___________
15. ___________
16. ___________
17. ___________
18. ___________

Student Identification Code: ________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in January 2004.
Modern Physics

Do three out of six problems

13) DATA
For a proton, \( mc^2 \equiv 10^9 \text{eV} \).
The diameter of our galaxy is \( D \equiv 10^5 \text{light-years} \).
The distance between the earth and the sun is about 8 light minutes.
One year is approximately \( 3 \times 10^7 \) seconds.
\[ c \equiv 3 \times 10^8 \frac{m}{s} . \]

a) A very energetic cosmic ray proton has an energy \( \varepsilon = 10^{19} \text{eV} \). In the reference frame of our galaxy, how long will it take this proton to cross the galaxy?

b) In the reference frame of this proton, how long will it take to cross our galaxy?

c) In the reference frame of the proton, what is the distance across the galaxy (in the direction of its motion)?

d) Compare the distance obtained in c) with the distance (in our reference frame) between the earth and the sun.

e) A photon and the high energy proton enter one side of our galaxy at the same time. They start at the same place and move in the same direction. Which will cross the galaxy first? The photon or the proton?

f) After they cross the galaxy, what will be the distance between the photon and the proton? (in our reference frame)
Modern Physics

Do three out of six problems

14) The strong force is invariant under rotations in isospin space, and hence isospin is a conserved quantity in strong interactions.

The nucleons (proton \( p \) and neutron \( n \)) are members of an isospin doublet;

\[
p \approx |1/2, 1/2\rangle \\
n \approx |1/2, -1/2\rangle
\]

The pions are members of an isospin triplet;

\[
\pi^+ \approx |1, 1\rangle \\
\pi^0 \approx |1, 0\rangle \\
\pi^- \approx |1, -1\rangle
\]

At some fixed energy, the cross section for the process

\[
\pi^+ + p \rightarrow \pi^+ + p
\]

is \( \sigma_0 \).

Suppose that in pion-nucleon interactions, the isospin 3/2 amplitude is much greater than the isospin 1/2 amplitude. Express the cross sections for the following processes (at the same energy) in terms of \( \sigma_0 \)

\[
\pi^- + p \rightarrow \pi^- + p \\
\pi^+ + n \rightarrow \pi^0 + p \\
\pi^0 + p \rightarrow \pi^+ + n
\]

NOTES:
Each of the nucleon-pion pairs is a linear combination of isospin 3/2 and isospin 1/2 amplitudes. Ignore the isospin 1/2 contribution to the scattering.

The Clebsch-Gordon table can be used to obtain the appropriate amplitudes. In this table, the two numbers on the left-hand column represent the \( z \)-component of the pion and nucleon isospins. The boxed numbers are the squared amplitudes (with signs outside the square root) and the pair of numbers above the boxes gives the total isospin and its \( z \)-component. For example, reading from the right-hand column of the third box,

\[
|1/2, -1/2\rangle = \frac{1}{\sqrt{3}} |1, 0\rangle |1/2, -1/2\rangle - \frac{2}{\sqrt{3}} |1, -1\rangle |1/2, 1/2\rangle
\]

20
Clebsch Gordon coefficients for 1x1/2

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Do three out of six problems

15) 
   a) Describe an experimental method to achieve the following Kelvin temperatures
      i) $4K$
      ii) $10^{-3}K$
      iii) The lowest temperature yet obtained

   b) What is the difference between a first-order and a second-order phase transition?

   c) Assume the temperature in the deep ocean is $4^\circ C$ and the temperature at the ocean
      surface is $20^\circ C$. Why is it probably impractical to use this temperature difference to
      generate electricity?

   d) Why does perspiration help keep you comfortable in hot dry weather?
Do three out of six problems

16) A one-dimensional crystal of identical atoms (each with one electron) is shown below.

a) Draw the reciprocal lattice of the above structure. Indicate its orientation and dimensions. Mark on the lattice the first, second and third Brillouin zones.

b) Sketch the dispersion relations for the energy $\varepsilon$ as a function of the wave number $k$ for nearly-free electrons in the extended zone scheme.

c) Sketch the frequency $\omega$ as a function of $k$ for phonons in the extended zone scheme.

d) Identify the similarities and differences between the curves obtained in b) and c).

e) Characterize the behavior of electrons and phonons at the zone boundaries, and give a physical reason for this behavior.

f) Is this linear chain of atoms an electrical conductor or insulator? Explain why.

g) Use the dispersion curves of b) and c) and appropriate conservation laws to determine if both normal and umklapp processes are possible for low temperature electron-phonon scattering.

h) How do normal and umklapp processes affect the electrical conductivity?
Do three out of six problems

17)

a) What is the approximate radius of the atomic nucleus of $^{208}$Pb? Describe one experimental method which could measure the size of such an atomic nucleus.

b) The stable isotopes of oxygen are $^{16}$O, $^{17}$O, and $^{18}$O. How would you expect the nucleus of $^{19}$O to decay? In your answer, list all the decay products.

c) The low lying states of $^{18}$O (8 protons and 10 neutrons) and $^{18}$Ne (10 protons and 8 neutrons) are similar. Explain what this suggests about the nuclear force. What are such pairs of nuclei called (i.e. where neutron number and proton number are exchanged)? Give one reason the states in $^{18}$O and $^{18}$Ne are not expected to be identical.

$^{208}$Pb is a "doubly magic closed shell nucleus" (82 protons and 126 neutrons). Adding one proton or one neutron forms the nuclei of $^{209}$Bi or $^{209}$Pb, respectively. The spectra for these two nuclei are shown below.

d) What low lying spin states would you expect for the nucleus formed by adding 2 protons to $^{208}$Pb ($^{210}$Po)?

e) What low lying spin states would you expect for the nucleus formed by adding 1 proton and 1 neutron to $^{208}$Pb ($^{210}$Bi)?
Do three out of six problems

18) Rutherford backscattering spectrometry is particularly useful because it is based primarily on two-body kinematics and the Rutherford cross section.

a) Consider the 180° elastic back-scattering of a projectile of energy $E_{beam}$ and mass $m_1$ from a target of mass $m_2$. Calculate, starting from basic principles, the energy of the projectile (mass $m_1$) backscattered to 180° from the target mass $m_2$.

b) The Rutherford scattering cross section for scattering a projectile of charge $q$ and energy $E$ from a target nucleus of charge $Q$ is

$$\sigma_R = \frac{Q^2q^2}{16E^2} \frac{1}{\sin^4 \theta}$$

Calculate $\sigma_R$ for 2MeV $^{++}$He ions scattering from gold ($Z=79$), scattering to 180°.

HINT:

$$\frac{Qq}{4E} = \frac{Z_1Z_2}{4} \left( \frac{e^2}{2a_o} \right) \left( \frac{2a_o}{E} \right)$$

where

$$\frac{e^2}{2a_o} \equiv 13.6eV$$

is the binding energy of the hydrogen atom ground state and $a_o = 0.53 \times 10^{-8} cm$ is the Bohr radius.

c) Consider a gold target with thickness $= 10^{-8} m$. Calculate the thickness of this gold target in atoms/cm$^2$.

HINT: The molecular weight of gold =179, the density of gold is 19.3gm/cm$^3$ and Avagadro’s number is $6.02 \times 10^{23}$.

d) Using the cross section calculated in b) and the target thickness calculated in c), assume you bombard this target with $10^{14}$ He ions (2MeV) and your detector has a solid angle of $2 \times 10^{-3} sr$, how many ions are expected to backscatter into your detector. What is the random statistical variation expected in a measurement of the number of backscattered ions?
DEPARTMENT OF PHYSICS

University at Albany
State University of New York

Comprehensive Field Examination

Part V

Saturday, January 24, 2004
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the three books at the end of the exam.

Quantum Mechanics

19. ______
20. ______
21. ______
22. ______

Student Identification Code: ________________________________

NOTE: This same code is to be used on all sections of the Comprehensive Examination taken in January 2004.
19) Shown below are three quantum mechanical energy barriers. Each has width \( d \) and a height \( V \). A particle with energy \( E = V / 2 \) is incident on each of these barriers from the left.

a) Sketch the wave function inside each barrier.

b) Compare the reflection coefficients for the two triangular barriers.

c) Can the reflection coefficient for the rectangular barrier be smaller than the reflection coefficient for either of the triangular barriers?
The double barrier shown below consists of two rectangular barriers separated by a distance \( w \). The reflection coefficient for the double barrier is periodic in the separation \( w \). That means the reflection coefficient is unchanged if \( w \to w + \Delta \). Find the minimum value of \( \Delta \).
Quantum Mechanics

Answer three of four questions

20) A particle moving in one dimension is described by the wave function

\[ \psi(x) = \frac{1}{\pi^{1/4} \sqrt{d}} \exp \left( ikx - \frac{x^2}{2d^2} \right) \]

a) Sketch the probability density \( \rho(x) = \psi^* (x) \psi(x) \) as a function of \( x \). Label the points where \( x = \pm d \).

b) Find the uncertainty in the position:

\[ (\Delta x) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \]

c) Find the uncertainty in the momentum:

\[ (\Delta p) = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \]

d) Find

\[ (\Delta x)(\Delta p) \]

NOTE:

\[ \int_{-\infty}^{\infty} \exp(-\alpha x^2) \, dx = \sqrt{\frac{\pi}{\alpha}} \]

Other useful identities can be obtained by differentiating the above expression with respect to \( \alpha \).
Answer three of four questions

21)

a) The spin part of a spin-1/2 particle’s wave function is

\[ \chi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \]

The Pauli matrices are:

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} : \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

At \( t = 0 \)

\[ \alpha(0) = \exp(-i\phi)\sin(\delta) \]
\[ \beta(0) = \exp(-i\phi)\cos(\delta) \]

Find the expectation value of this particle’s spin (at \( t = 0 \)) along the \( x, y \) and \( z \) directions.

b) This particle has an intrinsic magnetic moment \( \mu_0 \). It moves in a time-varying but spatially uniform magnetic field \( \vec{B}(t) \) whose components are:

\[ B_x = B\cos(\omega t) : \quad B_y = B\sin(\omega t) : \quad B_z = 0 \]

The spin dynamics are governed by the Hamiltonian

\[ H = -\mu_0 \vec{B}(t) \cdot \vec{\sigma} \]

where the components of \( \vec{\sigma} \) are the Pauli matrices.

Use the time-dependent Schrodinger equation to obtain a second-order differential equations for \( \alpha(t) \).

c) The general solution for \( \alpha(t) \) from part b) is

\[ \alpha(t) = C_1 \exp \left( -i \frac{E_1 t}{\hbar} \right) + C_2 \exp \left( -i \frac{E_2 t}{\hbar} \right) \]

Find \( E_1 \) and \( E_2 \).

d) What is the physical significance of \( E_1 \) and \( E_2 \) in the limit \( \omega \to 0 \)?
Quantum Mechanics

Answer three of four questions

22) An electron's physics is governed by the Hamiltonian

\[ H = \frac{p^2}{2m} + V(\vec{r}) \]

where \( \vec{p} \) is the momentum operator. Normalized eigenstates of this Hamiltonian are denoted \( |n\rangle \), with

\[ H |n\rangle = E_n |n\rangle \]

The oscillator strength \( f_{n,0} \) for the transition from an eigenstate \( |n\rangle \) to the ground state \( |0\rangle \) is

\[ f_{n,0} = \frac{2m}{3\hbar^2} (E_n - E_0) |\langle n|\vec{r}|0\rangle|^2 \]

where

\[ |\langle n|\vec{r}|0\rangle|^2 = |\langle 0|x|n\rangle\langle n|x|0\rangle + \langle 0|y|n\rangle\langle n|y|0\rangle + \langle 0|z|n\rangle\langle n|z|0\rangle |^2 \]

a) Show that

\[ \langle n|\vec{p}|0\rangle = \frac{im}{\hbar} (E_n - E_0) \langle n|\vec{r}|0\rangle \]

b) Using the result of part a) show that

\[ \sum_n f_{n,0} = 1 \]

HINTS:
For part a), consider the commutator of the Hamiltonian with the components of \( \vec{r} \).
Remember that \([x, p_x] = i\hbar\), with similar results for the other components of the position and momentum.

For part b), assume the states \( |n\rangle \) form an orthonormal complete set so \( \sum_n |n\rangle\langle n| = 1 \).
DEPARTMENT OF PHYSICS

University at Albany
State University of New York

Comprehensive Field Examination

Part VI

Saturday, January 24, 2004
1:00 - 3:00 PM

Instruction:

**Answer any two out of three questions**

Please check below the two problems you have done. Do not place your name on the examination booklet or this cover sheet. Each problem SHOULD be done in a separate examination book. The problems within each area carry equal weight.

Turn in the cover sheet and the two books at the end of the exam.

**Advanced Quantum Mechanics**

23. ________

24. ________

25. ________

Student Identification Code: __________________________________________

**NOTE:** This same code is to be used on all sections of the Comprehensive Examination taken in January 2004.
Answer two out of three questions

23) Two spin $\frac{1}{2}$ nucleons interact with a potential

$$V(r) = \frac{A}{r} \exp\left(-\frac{r}{B}\right) \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

where $A$ and $B$ are positive constants and $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin operators for the two nucleons. Before scattering, the spin of nucleon #1 is “up” and the spin of nucleon #2 is “down”.

a) Find the Born approximation for the center of mass differential cross section when the two nucleons are a proton and a neutron.

b) Find the Born approximation for the center of mass differential cross section when the two nucleons are two protons.

NOTES:
When a single particle of mass $m$ scatters from a spherically symmetric potential $V(r)$, the differential cross section is the square of the scattering amplitude.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

where $\theta$ is the scattering angle.

In the Born approximation, the scattering amplitude is

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int \exp\left(i(\vec{k} - \vec{k}') \cdot \vec{r}\right) V(r) d^3r$$

where $\vec{h}\vec{k}$ is the momentum of the incident particle, and $\vec{h}\vec{k}'$ is the momentum of the scattered particle.

When distinguishable particles scatter from each other, the same expressions apply with the following modifications:
1) The cross sections are given in the center of mass coordinate system
2) The momenta $\vec{h}\vec{k}$ and $\vec{h}\vec{k}'$ refer to the differences of the momenta of the two particles.
3) The mass $m$ is replaced by the reduced mass $\mu$.

When two identical particles scatter from each other, there are additional modifications.
1) The cross sections refer to the probability of seeing either particle.
2) If the spatial part of the wave function does not change sign when the particles are exchanged,

$$\frac{d\sigma}{d\Omega} = |f(\theta) + f(\pi - \theta)|^2$$

3) If the spatial part of the wave function does change sign when the particles are exchanged,

$$\frac{d\sigma}{d\Omega} = |f(\theta) - f(\pi - \theta)|^2$$
An integral of the form
\[ \int \exp (i \vec{p} \cdot \vec{r}) V(r) d^3 r \]
can be simplified in spherical coordinates. Integrate over the angles first, and use
\[ \vec{p} \cdot \vec{r} = pr \cos(\theta) \]
The spin wave function for particle #1 with spin up and particle #2 with spin down can be decomposed into a sum of spin-singlet and spin-triplet states.
\[ |\chi\rangle = \frac{1}{\sqrt{2}} (|\text{singlet}\rangle + |\text{triplet}\rangle) \]
A spin-singlet state changes sign when the two spins are interchanged. A spin-triplet state does not change sign when the two spins are interchanged.

Also,
\[ \vec{\sigma}_1 \cdot \vec{\sigma}_2 |\text{singlet}\rangle = -3 |\text{singlet}\rangle \]
and
\[ \vec{\sigma}_1 \cdot \vec{\sigma}_2 |\text{triplet}\rangle = |\text{triplet}\rangle \]
Advanced Quantum Mechanics

Answer two out of three questions

24) A two-dimensional harmonic oscillator is described by the Hamiltonian

\[ \hat{H} = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1 \right) \]

where \( \hat{a}^\dagger, \hat{a}, \hat{b}^\dagger \) and \( \hat{b} \) obey the commutation relations

\[ [\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1 \]

and

\[ [\hat{a}, \hat{b}^\dagger] = [\hat{b}, \hat{a}^\dagger] = [\hat{a}, \hat{b}] = [\hat{b}^\dagger, \hat{a}^\dagger] = 0 \]

These operators raise or lower the number states \( |n_x, n_y\rangle \) \((n_x = 0, 1, 2, \cdots; n_y = 0, 1, 2, \cdots)\) as follows:

\[ \hat{a}^\dagger |n_x, n_y\rangle = \sqrt{n_x + 1} |n_x + 1, n_y\rangle \quad \hat{b}^\dagger |n_x, n_y\rangle = \sqrt{n_y + 1} |n_x, n_y + 1\rangle \]

\[ \hat{a} |n_x, n_y\rangle = \sqrt{n_x} |n_x - 1, n_y\rangle \quad \hat{b} |n_x, n_y\rangle = \sqrt{n_y} |n_x, n_y - 1\rangle \]

a) Obtain the energy eigenvalues \( E_n \) \((n = n_x + n_y)\) given by

\[ \hat{H} |n_x, n_y\rangle = E_n |n_x, n_y\rangle \]

b) What is the degeneracy of the \( n' \)th eigenvalue?

c) The oscillator is subjected to a small perturbation

\[ V = \lambda \hat{x} \hat{y} \]

where

\[ \hat{x} = \sqrt{\frac{\hbar}{2m \omega}} (\hat{a} + \hat{a}^\dagger) \quad \hat{y} = \sqrt{\frac{\hbar}{2m \omega}} (\hat{b} + \hat{b}^\dagger) \]

Calculate the first order (in \( \lambda \)) corrections to the energies of all the states whose unperturbed energy is \( E_1 \) (first excited energy level).

d) Find the normalized eigenstates corresponding to the energy levels obtained in part c). Express these states in terms of the states \( |n_x, n_y\rangle \).
Answer two out of three questions

25) An electron with mass \( m \) and charge \(-e\) moves as a free particle on a circle of radius \( R \). The position of the electron on the ring is specified by the angle \( \phi \) and the electron's wave function can be written as \( \psi(\phi) \). Then the Schrodinger equation for this electron is

\[
\frac{1}{2mR^2} p_\phi^2 \psi(\phi) = E \psi(\phi)
\]

where

\[
p_\phi = -i\hbar \frac{d}{d\phi}
\]

is the operator corresponding to the angular momentum component normal to the ring. For all parts of this problem, apply the boundary condition \( \psi(\phi) = \psi(\phi + 2\pi) \).

a) Find the energy levels and their degeneracy when there is no magnetic field.

This ring is placed around a solenoid with radius \( a < R \). (See figure on next page.) Current in this solenoid produces a magnetic field inside the solenoid. The vector potential associated with this field (for \( r > a \)) can be taken to be in the \( \hat{\phi} \) direction with

\[
A_\phi(r) = \frac{\Phi}{2\pi r}
\]

where \( \Phi \) is the magnetic flux in the solenoid. The vector potential changes the Schrodinger equation. One must make the replacement

\[
\frac{1}{R} p_\phi \rightarrow \left( \frac{1}{R} p_\phi - (-e)A_\phi(R) \right)
\]

b) Find the new energy levels of this system. Which values of the magnetic flux \( \Phi \) produce degenerate energy levels? Which values of the magnetic flux \( \Phi \) produce a degenerate ground state?

The electric current around the ring which is produced by an electron in a state \( \psi(\phi) \) is

\[
\langle I \rangle = \frac{-e}{m(2\pi R)} \int_0^{2\pi} \psi^*(\phi) \left( \frac{1}{R} p_\phi - (-e)A_\phi(R) \right) \psi(\phi) d\phi
\]

assuming the state is normalized by the condition

\[
\int_0^{2\pi} \psi^*(\phi)\psi(\phi) d\phi = 1
\]

c) Find the current produced by an electron in the ground state of this system as a function of the magnetic flux, \( \Phi \).