DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part I

Tuesday, September 3, 2002
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Classical Physics

1. __________
2. __________
3. __________
4. __________

Student Identification Code: __________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2002.
Classical Physics

Fall 2002

Answer Three (3) out of Four (4) Questions

1. A ball of mass $m$ is dropped from rest off a high bridge. As it falls, there is a drag force acting on the ball given by $F_d = -bv$ where $v$ is the speed of the ball.

   (a) Calculate the speed $v(t)$ of the ball as a function of time. What is the terminal speed $v(t \to \infty)$ reached by the ball?

   (b) Calculate the distance $x(t)$ the ball has fallen as a function of time.

   (c) Show that your answer to part (b) agrees with the result for a freely falling body in the limit that $b \to 0$.

   (d) If, 10 seconds after being released the ball has reached 90 percent of its terminal speed, so that

   \[ v(t = 10 \text{ s}) = 0.90v(t = \infty). \]

   How long will it take the ball to reach 99 percent of its terminal speed? That is, find the $t^*$ for which

   \[ v(t^*) = 0.99v(t = \infty). \]
Answer Three (3) out of Four (4) Questions

2. (a) Consider two identical dominoes (a domino is a rectangular solid of uniform density) of mass \( m \), width \( w \) and length \( L \) on a horizontal surface. The first is leaning against the second, which is standing vertically. What is the minimum horizontal spacing \( a \) between the dominoes such that the second domino will topple? Assume no friction between the dominoes, and sufficient friction between each domino and the horizontal surface that they do not slip. Take \( w = L/5 \).

Hint: Start by finding the horizontal position of the center of mass of the leaning domino relative to its pivot point.

(b) Now suppose that the second domino is standing at the edge of a step of height \( h \). If the horizontal spacing \( a \) between the dominoes is \( L/2 \) what is the maximum height \( h \) such that the second domino will still topple? Again, take \( w = L/5 \). (A series of dominoes on steps will allow the line of dominoes to "fall uphill.")
Answer Three (3) out of Four (4) Questions

3. A uniform line charge \( \lambda \) is placed on an infinite straight wire, a distance \( d \) above a grounded conducting plane. (The wire runs parallel to the \( x\)-axis and directly above it, and the conducting plane is the \( xy\)-plane.)

(a) Find the potential in the region above the plane.

(b) Find the charge density \( \sigma(x, y) \) induced on the conducting plane.

(c) Find the force per unit length between the line charge and the conducting plane. Is the force attractive or repulsive?
Answer Three (3) out of Four (4) Questions

4. (a) Consider an infinitely long cylindrical conductor of radius $b$ whose axis coincides with the $z$-axis. A steady current $I$ flows uniformly inside the conductor along the positive $z$-direction. Assume for simplicity that the permeability of the conductor is $\mu_0$. Find the current density $\mathbf{J}$ and the $x$- and $y$-component of the magnetic induction $\mathbf{B}(\mathbf{r})$ inside the conductor.

(b) Now consider an infinitely long cylindrical conductor of radius $b$ which contains in it an infinitely long cylindrical cavity of radius $a$ ($a < b - d$) whose axis is parallel to the conductor's axis. Again assume that the conductor's axis coincides with the $z$-axis. The distance between the conductor and the cavity axis is $d$ as shown in the figure below. A steady current $I$ flows uniformly inside the conductor along the positive $z$-direction. Assume for simplicity that the permeability of the conductor is $\mu_0$. Find the magnetic induction $\mathbf{B}(\mathbf{r})$ inside the cavity.

Hints: Use principle of superposition. Express your results for $\mathbf{B}(\mathbf{r})$ either in terms of Cartesian coordinates or by replacing the respective unit vectors in the $\varphi$-direction with cross products of the other two unit vectors.
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part II

Tuesday, September 3, 2002
1:00 - 3:00 PM

Instruction:

Answer one from Questions 5 and 6 and one from Questions 7 and 8

Please check below the two (2) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Advanced Classical Physics

5. __________
6. __________
7. __________
8. __________

Student Identification Code: __________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2002.
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

5. (a) A "wheel" consists of a hollow cylinder of mass $M$ and an axle of mass $m$. The hollow cylinder has radius $R$ and moment of inertia $I = MR^2$. The axle has mass $m$ and negligible radius. This wheel is placed on a $\theta = 30^\circ$ incline ($\sin 30^\circ = 1/2$). It rolls without slipping down this incline. Let $s(t)$ be the distance the wheel moves along the incline as a function of the time $t$. Assume the system starts at rest with $s(0) = 0$. Find the acceleration $a(t)$, and $s(t)$ for $t > 0$.

(b) The axle is replaced by a massless drum of radius $b = (3/4)R$ which extends over the edge of the inclined plane. A massless string is wound around the drum (in the direction shown in the Figure). A mass $m$ hangs down vertically from the edge of the drum. Using the constraints between $s$, $\theta$, and $x$ write down the Lagrangian for this system and find the equation of motion for $s(t)$. Assume that the motion of the wheel takes place in the plane of the Figure and that the string remains vertical throughout the motion. For what mass ratio $m/M$ would the wheel be stationary?
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

6. Consider the arrangement shown in the Figure below. Spring #1 is attached to a fixed wall and stretches along the x-axis to a (massless) pivot point $P$. Spring #2 extends at a $45^\circ$ angle from $P$ to a mass $M$. Both springs have negligible mass and spring constant $k$. The arrangement is horizontal, hence, neglect gravity.

(a) Set up the Lagrangian in terms of the spring extensions $x_1$ and $x_2$ and obtain the two coupled equations of motion.

(b) Find the normal frequencies and the normal modes.

(c) Find the Hamiltonian for the system.

(d) Find Lagrange's equations of motion via Hamilton's canonical equations. Compare with part (a).
Advanced Classical Physics

Fall 2002

Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

7. A beam of x rays described by the plane wave \( E_0 e^{(k_x r - \omega t)} \) is incident on a material sample and induces a polarization (dipole moment per unit volume)

\[
P(r)e^{-i\omega t} = -\frac{e^2}{m \omega^2} N(r) E_0 e^{(k_x r - \omega t)},
\]

where \( N(r) \) is the electron density of the sample and \( e \) and \( m \) are the charge and mass of the electron, respectively.

(a) Calculate the scattered electric field \( E(t, r) \) far from the sample. Show that it is proportional to the Fourier transform of the electron density. What is the proportionality constant?

(b) If the sample is a crystal its periodic electron density is described by a Fourier series

\[
N(r) = \sum_H N_H e^{i\mathbf{H} \cdot \mathbf{r}},
\]

where \( \mathbf{H} \) are the reciprocal lattice vectors (i.e., \( H = 2\pi d \) where \( d \) are the interplanar spacings). Show that in this case the scattered field \( E(t, r) \) satisfies Bragg's law. That means the angle between the incident (reflected) beam and the crystal plane, \( \theta \), is given by \( \sin \theta = \lambda / (2d) \).

Hints: The electric field at a distant point \( r \) radiated by a dipole \( \mathbf{P}(r')dV' \) located in a small volume element \( dV' \) at \( r' \) is

\[
dE(r) = -\mathbf{k} \times (\mathbf{k} \times \mathbf{P}(r')dV') \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{|r-r'|}.
\]

Far from the sample you may use \( |r-r'| \approx r - \hat{r} \cdot r' \) where \( \hat{r} = r / r \).
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

8. (a) A charge $Q$ is placed at point $a$ on the $z$-axis. Write down the multipole expansion for the potential $V(r, \theta)$ for $r > a$. Identify the monopole and dipole terms.

(b) The charge is now surrounded by a conducting spherical shell of inner radius $R$, outer radius $b$ centered at the origin as shown in the figure. The shell is held at potential $V_0 = 0$. Find the location and value of the image charge $q$ (see Figure) needed to fulfill the boundary condition at $R$ for all values of $\theta$. (Hint: Use the multipole expansion for the image charge also.)

(c) Give the solution for the potential of part (b) for $r < R$ and find the induced surface charge distribution at $r = R$. Indicate the magnitude and polarity of the surface charge in a sketch. What is the potential for $r \geq R$?

(d) Suppose the conducting spherical shell surrounding the charge $Q$ is not grounded and carries no net charge. What, if anything changes in your results to parts (b) and (c) and what would be the potential for $b > r \geq R$ and for $r \geq b$?
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part III

Thursday, September 5, 2002
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Statistical and Thermal Physics

9. __________
10. __________
11. __________
12. __________

Student Identification Code: __________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2002.
Answer Three (3) out of Four (4) Questions

9. Two identical bodies, each with heat capacity at constant pressure $C$, are used as heat reservoirs for an engine. Initially their temperatures are $T_1$ and $T_2$; finally, as a result of the operation of the engine, they attain a common final temperature $T_f$. Assume that $C$ is independent of temperature and that the bodies remain at constant pressure.

   (a) Calculate the total work $W$ done by the engine in terms of $C$, $T_1$, $T_2$ and $T_f$.

   (b) Calculate the total entropy $\Delta S$ generated in the process in terms of $C$, $T_1$, $T_2$ and $T_f$. Use this to derive an inequality relating $T_f$ to $T_1$ and $T_2$.

   (c) What is the maximum amount of work that the engine might perform for given initial temperatures $T_1$ and $T_2$?
Answer Three (3) out of Four (4) Questions

10. The internal energy $U$ of an ideal gas depends only on temperature $T$, and

\[ \frac{dU(T)}{dT} = C_v, \]

where $C_v$ is the constant-volume heat capacity. That means for small changes in pressure, $P$, and volume, $V$, the corresponding change in the internal energy is

\[ dU = C_v \left( \frac{\partial T}{\partial P} \right)_V dP + \frac{\partial T}{\partial V} \bigg|_P dV. \]

(a) Write down the equation of state relating pressure, volume and temperature for $n$ moles of an ideal gas. Use this equation of state to write the above expression for $dU$ in terms of $C_v$, $nR$, $P$, $V$ as well as $dP$ and $dV$. Use the symbol $R$ for the universal (molar) gas constant.

(b) The internal energy can also be written in terms of the entropy $S$ and the volume $V$, and the resulting formula for $dU$ is an expression of the first law of thermodynamics. Write down this formula.

(c) Equate the two formulas for $dU$ for the case of an adiabatic expansion of an ideal gas, and use the result to derive the equation

\[ \left. \frac{\partial P}{\partial V} \right|_s = -\gamma \frac{P}{V}. \]

(d) Express $\gamma$ in terms of $C_v$ and $nR$.

(e) Using the relation derived in (c), find $\gamma$ for the ideal gas CH$_4$ at 41 °C, if the velocity of sound in CH$_4$ at 41 °C is 466 m/s. The molar mass of CH$_4$ is 16.04 g/mol and the universal gas constant $R = 8.314 \text{ J/mol} \cdot \text{K}$. Note that the velocity of sound $v_s$ is related to the bulk modulus $B$ and the density $\rho$ of CH$_4$ according to:

\[ v_s = \sqrt{\frac{B}{\rho}} \quad \text{and} \quad B = -V \left. \frac{dP}{dV} \right|_s. \]
Answer Three (3) out of Four (4) Questions

11. Bose-Einstein condensation of a fluid occurs when the de Broglie wavelength of a "typical" particle becomes greater than the average nearest-neighbor distance. One can interpret the momentum in the de Broglie equation as

\[ p = \sqrt{\langle p^2 \rangle} \]

where the \( \langle p^2 \rangle \) means thermal average for a single particle.

(a) Write down the de Broglie equation and use the above to obtain an approximate expression for the Bose-Einstein condensation temperature \( T_B \) in terms of the particle density \( (N/V) \), the particle mass \( m \), Boltzmann's constant \( k \) and Planck's constant \( h \).

(b) For temperatures \( T \) less than the transition temperature, \( T_B \), the fraction \( f \) of particles in the ground state \( (f = N(0)/N) \) is a number between zero and one. Find the chemical potential \( \mu \) to order \( 1/N \) as a function of \( T \) when \( T \ll T_B \). Assume the ground state energy is zero.

(c) Bose-Einstein condensation occurs for liquid helium at about 2 K. Since 1995, Bose-Einstein condensation has also been observed in dilute gasses. Compare the density of a Bose-Einstein dilute gas to the density of liquid helium. Make the following assumptions:

1. The transition temperature of the dilute gas is lower than the transition temperature of liquid helium by a factor of \( 10^7 \).
2. The gas atoms are 10 times as heavy as a helium atom.
12. A system contains $N$ sites and $N$ electrons. At each site there is only one accessible orbital which can be occupied by zero, one or two electrons (of opposite spin). The site energy is zero if the site is either empty or singly occupied and it is $\varepsilon$ if it is doubly occupied (there is some repulsion between the electrons). In addition there is an external magnetic field $B$ which acts on the spin coordinates. The energy of a magnetic dipole aligned with $B$ is $-\mu B$.

(a) Write down the grand canonical partition function.

(b) Calculate the chemical potential from the condition that on the average there is one electron per site.

(c) Calculate the mean magnetization $m$ per site.

(d) Calculate the susceptibility $\chi = \left( \frac{\partial m}{\partial B} \right)_T$ for small $B$. 
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part IV

Thursday, September 5, 2002
1:00 - 3:00 PM

Instruction:

Answer any three out of six questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Modern Physics

13. 
14. 
15. 
16. 
17. 
18. 

Student Identification Code: 

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2002.
Modern Physics

Answer three (3) out of six (6) Questions.

13. In the Bohr model, the energy levels of the hydrogen atom are given by the formula

\[ E_n = -\frac{hR}{n^2} \quad n = 1, 2, 3, \ldots, \]

where \( h \) is Planck's constant and \( R \) is the Rydberg constant.

The Bohr correspondence principle states that for orbits of sufficiently high principal quantum number \( n \) the frequency, \( f_Q \), of the quantum mechanical transition \( n \rightarrow n \pm 1 \) becomes equal to the frequency, \( f_C \), of the electron motion along the classical orbit. Use this correspondence principle to derive the expression for the Rydberg constant in terms of the fundamental constants \( h \), \( e \), and \( m \), where \( e \) and \( m \) are the electron charge and mass.

Hint: Express both frequencies in terms of the energy of the orbit.
Modern Physics

Fall 2002

Answer three (3) out of six (6) Questions.

14. (a) A beam of relativistic protons collides with protons in a stationary target. What is the threshold kinetic energy for an incident proton colliding with a proton at rest to create a proton-antiproton pair via the reaction $p + p \rightarrow p + p + p + \bar{p}$? You may take $mc^2 = 1 \text{ GeV}$ for the proton rest mass.

(b) Suppose that instead of the stationary target, the beam of protons collides head on with another, counter propagating beam of protons with the same energy. What is the threshold kinetic energy (of one proton) for creation of a proton-antiproton pair? Compare with your result for part (a).
Modern Physics

Answer three (3) out of six (6) Questions.

15. The quantum numbers for the various quarks are tabulated below:

<table>
<thead>
<tr>
<th>Quark</th>
<th>B</th>
<th>Q</th>
<th>I</th>
<th>$I_z$</th>
<th>s</th>
<th>c</th>
<th>b</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>1/3</td>
<td>+2/3</td>
<td>1/2</td>
<td>+1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>1/3</td>
<td>−1/3</td>
<td>1/2</td>
<td>−1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>s</td>
<td>1/3</td>
<td>−1/3</td>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>1/3</td>
<td>+2/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>1/3</td>
<td>−1/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>1/3</td>
<td>+2/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Here $B =$ baryon number, $Q =$ charge, $I, I_z =$ isospin and its $z$ component, $s =$ strangeness, $c =$ charm, $b =$ bottom, $t =$ top. From SU(3) symmetry, the spin $J = (1/2)^+ \ baryon$

multiplet consists of eight members:

- $p, n$ with strangeness $s = 0$,
- $\Lambda^0, \Sigma^-, \Sigma^0, \Sigma^+$ with strangeness $s = -1$,
- $\Xi^-, \Xi^0$ with strangeness $s = -2$.

The $\Omega^-$ with strangeness $-3$ does not belong to spin $J = (1/2)^+$ octet.

Fill in the table below for the eight members of the $J = (1/2)^+ \ baryon$ multiplet and draw its isospin versus strangeness chart indicating the positions of the eight multiplet members. Note $c = b = t = 0$ for all eight members.
16. Rutherford backscattering spectrometry (RBS) is perhaps the most commonly practiced method of ion beam analysis. The method involves bombarding a sample with incident ions (commonly 2 MeV He\(^+\)) and recording the number and energy of ions backscattered from the sample.

(a) Describe how 2 MeV He\(^+\) ions are obtained in the laboratory. Include in your answer how the ions are accelerated, how they are "purified" (i.e. how other ions are removed from the ion beam), and give (approximately) the uncertainty in the ion energy.

(b) RBS involves measuring the backscatter yield per incident ion. Describe how the number of He\(^+\) ions incident on a sample is measured. What limits the accuracy of this measurement? What is a typical uncertainty in this measurement in a routine RBS measurement?

(c) Most RBS measurements are made with a "Si detector". Describe briefly how such a detector works. What limits the energy resolution of such a detector? Why does the "noise level" in such a detector decrease when the voltage bias is applied to the detector? What is the detection efficiency for a 2 MeV He\(^+\) ion that is incident on the active surface of such a detector?

(d) What problems occur if you run too large an incident He beam?

(e) Assume you have a sample that consists of approximately a monolayer of a single (unknown) heavy element on a light substrate. Describe how using RBS you would determine what element was present. How would you determine how much of that element was present?
Modern Physics

Fall 2002

Answer three (3) out of six (6) Questions.

17. . (a) Some element A has a linear absorption coefficient $\mu_A = 100 \text{ cm}^{-1}$ for x-rays of a given energy. What thickness material is needed to absorb 50 percent of the x-rays?

(b) If the density of the material is $\rho_A = 2 \text{ g/cm}^3$, what is its mass absorption coefficient $\mu_{pA}$ (in cm$^2$/g)?

(c) What is the relation between $\mu_A$ and the absorption cross section $\sigma_A$? Between $\mu_{pA}$ and $\sigma_A$?

(d) The mass absorption coefficients of aluminum (Al, atomic mass $A_{Al} = 27$) and oxygen (O, atomic mass $A_O = 16$) for copper $K\alpha$ x-rays are 50 and 11 cm$^2$/g, respectively. Find the mass absorption coefficients of Al$_2$O$_3$. The procedure you need to use is referred to as Bragg’s rule.
Modern Physics  

Fall 2002

Answer three (3) out of six (6) Questions.

18. So-called “Cooper pairs” play a central role in the description of superconductors.

(a) What are the particles that make up a Cooper pair?

(b) What is the charge of a Cooper pair?

(c) Briefly describe an experiment which can measure the charge of a Cooper pair.

(d) Phonons are responsible for the binding of Cooper pairs in ordinary superconductors. Briefly describe an observation which demonstrates the role of phonons in determining the superconducting transition temperature.

(e) Estimate the binding energy of a Cooper pair for a superconducting transition temperature of 10 degree Kelvin ($kT \approx 1/40$ eV at room temperature).
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part V

Saturday, September 7, 2002
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Quantum Mechanics

19. 
20. 
21. 
22. 

Student Identification Code: 

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2002.
Quantum Mechanics

Answer 3 out of 4 problems

19. An atom is found in the orbital angular momentum state \( Y_1^1(\theta, \varphi) \), where

\[
L^2 Y^m_l(\theta, \varphi) = (l(l+1)\hbar^2) Y^m_l(\theta, \varphi),
\]

and

\[
L_x Y^m_l(\theta, \varphi) = m\hbar Y^m_l(\theta, \varphi).
\]

(a) What value will the measurement of \( L_x \) in the state \( Y_1^1(\theta, \varphi) \) give? With what probability will the value occur?

(b) If \( X(\theta, \varphi) = aX^1_l(\theta, \varphi) + bX^0_l(\theta, \varphi) + cX^{-1}_l(\theta, \varphi) \), show that \( X(\theta, \varphi) \) is an eigenstate of \( L^2 \). What is the corresponding eigenvalue?

(c) Suppose the state \( X(\theta, \varphi) \) above satisfies the eigenvalue equation

\[
L_x X_l(\theta, \varphi) = \lambda \hbar X_l(\theta, \varphi).
\]

What are the possible values for \( \lambda \)?

(d) Determine the coefficients \( a, b, c \) of \( X_l(\theta, \varphi) \) for each value of \( \lambda \) in such a way that

\[
|a|^2 + |b|^2 + |c|^2 = 1.
\]

(e) Express \( Y_1^1(\theta, \varphi) \) in terms of the eigenstates of \( L_x \) and find the probability that measurement of \( L_x \) in the state \( Y_1^1(\theta, \varphi) \) finds \( \lambda = 0 \).

Hint: Express \( Y_1^1(\theta, \varphi) \) using your results from part (d).

Note: \( L_x = \frac{1}{2}(L_+ + L_-) \), \( L_y = \frac{1}{2i}(L_+ - L_-) \), and \( L_x Y^m_l(\theta, \varphi) = \sqrt{(l+m)(l\pm m+1)}\hbar Y^{m+1}_l(\theta, \varphi) \).
Quantum Mechanics

Answer 3 out of 4 problems

20. A generic state of a spin-1 particle can be written as

\[ |\Psi\rangle = a|+1\rangle + b|0\rangle + c|-1\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \]

In the same basis \((|+1\rangle, |0\rangle, |-1\rangle)\) the Hamiltonian is given by

\[ \hat{H} = \begin{pmatrix} 0 & A & 0 \\ A & 0 & 0 \\ 0 & 0 & B \end{pmatrix}. \]

(a) Calculate the energy eigenvalues and the normalized eigenvectors.

(b) Calculate the state \(|\Psi(t)\rangle\) at time \(t\) if the initial state is \(|\Psi(0)\rangle = |-1\rangle\).

(c) If the initial state is \(|\Psi(0)\rangle = |+1\rangle\), what are the probabilities that at time \(t\) the particle will be found in each of the states \(|+1\rangle, |0\rangle, |-1\rangle\)?
Quantum Mechanics

Answer 3 out of 4 problems

21. Two spin = 1/2 distinguishable particles (e.g. a proton and a neutron) are in the spin-singlet state.

$$\chi(1,2)_{s=0} = \frac{1}{\sqrt{2}} (|+z\rangle_1 |-z\rangle_2 - |-z\rangle_1 |+z\rangle_2),$$

where $|+z\rangle_1$ means particle #1 has its spin aligned along the $+\hat{z}$ direction, and $|-z\rangle_2$ means particle #2 has its spin aligned along the $-\hat{z}$ direction.

(a) The spin of particle #1 along the z-axis is measured. It is found that the spin is parallel to $+\hat{z}$. Then the spin of particle #2 along the z-axis is measured. What is the probability that this second spin will be found parallel to $+\hat{z}$ and what is the probability that the spin will be found parallel to $-\hat{z}$?

(b) The experiment is repeated for the same initial singlet state. The spin of particle #1 along the z-axis axis is measured. Again, it is found that the spin is parallel to $+\hat{z}$. Then the spin of particle #2 along the x-axis is measured. What is the probability that the second spin will be found parallel to $+\hat{x}$ and what is the probability that the spin will be found parallel to $-\hat{x}$?

(c) The experiment is repeated for the same initial singlet state. This time the spin of particle #1 along the x-axis is measured. It is found that the spin is parallel to $+\hat{x}$. Then the spin of particle #2 along the x-axis is measured. What is the probability that the spin will be found parallel to $+\hat{x}$ and what is the probability that the spin will be found parallel to $-\hat{x}$?

(d) The experiment is repeated for the same initial singlet state. The spin of particle #1 along the x-axis is measured. It is found that the spin is parallel to $+\hat{x}$. Then the spin of particle #2 along the z-axis is measured. What is the probability that the spin will be found parallel to $+\hat{z}$ and what is the probability that the spin will be found parallel to $-\hat{z}$?
Quantum Mechanics

Answer 3 out of 4 problems

22. So called ultra-cold neutrons are neutrons with such low energy that they cannot penetrate many solid surfaces and, hence, they can be contained in "bottles" or set on "tables". Consider such an ultracold neutron in the gravitational potential resting on a table. Using the variational method with a trial wave function selected from below, evaluate the most probable height above the table for a neutron in the lowest energy state.

Trial wave functions: Which of the following is a physically sensible choice for a trial wave function ($\alpha$ is the variational parameter.): $\psi(z) = e^{-\alpha z}$ or $\psi(z) = ze^{-\alpha z}$? Choose one of these two for your calculation. Justify your choice.

Notes and Hints: Let $z = 0$ at the table surface. The potential $V = \infty$ for $z < 0$. This can be considered a one-dimensional problem.

$$\frac{\hbar^2}{2m_n} = 2 \times 10^{-19} \text{ eV cm}^2,$$

$$m_n g = 10^{-9} \text{ eV cm}^{-1},$$

where $m_n$ is the neutron mass and $g$ is the acceleration of gravity.

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$
DEPARTMENT OF PHYSICS

University at Albany
State University of New York

Comprehensive Field Examination

Part VI

Saturday, September 7, 2002
1:00 - 3:00 PM

Instruction:

Answer any two out of three questions

Please check below the two (2) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Advanced Quantum Mechanics

23. ________

24. ________

25. ________

Student Identification Code: ________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2002.
Answer Two (2) out of Three (3) Questions.

23. A particle of mass $m$ and charge $q$ moves on a ring of radius $a$. The ring lies in the $x$-$y$ plane and is centered at the origin. Positions along the ring are measured by the coordinate $s$. The angle along the ring is measured by $\theta$. (See Figure below.)

The wave function for this particle is $\psi(s)$ and (assuming no potential energy) the Hamiltonian is

$$\hat{H} = \frac{\hat{p}_s^2}{2m},$$

where the momentum operator for motion in the $s$-direction is $\hat{p}_s = -i\hbar \frac{\partial}{\partial s}$.

The operator for angular momentum along the $z$-axis is $\hat{l}_z = -i\hbar \frac{\partial}{\partial \theta}$.

(a) Find the normalized wavefunction solutions to the Schrödinger equation $\hat{H}\psi(s) = E\psi(s)$ which are also eigenfunctions of $\hat{l}_z$. Find the energy eigenvalues. What are the degeneracies of the ground, the first and second excited state? Find the eigenvalues of $\hat{l}_z$.

(b) A magnetic field $\mathbf{B} = B_z \mathbf{\hat{z}}$ is applied along the $z$-axis. The vector potential can be written as $\mathbf{A} = \gamma \mathbf{r} \times \mathbf{B}$. Working in Cartesian coordinates, find the constant $\gamma$ so that $\mathbf{B} = \nabla \times \mathbf{A}$. Compare the direction of $\mathbf{A}$ with the direction of the motion on the ring. What is the relation between the (constant) magnitudes $A$ and $B_z$ on the ring of radius $a$?

(c) The Hamiltonian in the magnetic field is modified by the replacement

$$\mathbf{p} \rightarrow (\mathbf{p} - q\mathbf{A}).$$

Find the wave functions and energy eigenvalues in the magnetic field. Qualitatively plot how the energy levels shift as compared to the energy levels of part (a). What values of the magnetic field will produce a degenerate ground state doublet?

Note: The wave functions must be single valued on the ring of radius $a$. That means $\psi(s + 2\pi a) = \psi(s)$

![Diagram of a ring with angle $\theta$ and position $s$.]
Answer Two (2) out of Three (3) Questions.

24. A particle of mass $m$ and charge $q$ is in a *three-dimensional* harmonic oscillator potential $V = k(x^2 + y^2 + z^2)/2$ with equal force constant $k$ in all three directions.

   (a) Write down the energy eigenvalues and the eigenfunctions for the four states of lowest energy. Is there any degeneracy?

   (b) At time $t = -\infty$ the oscillator is in its ground state. It is then perturbed by a spatially uniform time-dependent electric field

   $$ E(t) = A e^{-(t/\tau)^2} \hat{z}, $$

   where $A$ and $\tau$ are constants. Find the perturbation to the harmonic oscillator Hamiltonian and calculate in lowest-order time dependent perturbation theory the probability that the oscillator is in an excited state at $t = \infty$.
   
   *Hint:* Complete the square in the exponent and use the integral given below.

   (c) What is the probability for the harmonic oscillator to be left in the ground state?

   **Note:** For a *one-dimensional* harmonic oscillator in the coordinate basis ($\omega = \sqrt{k/m}$)

   $$ \psi_n(x) = \left(\frac{m \omega}{\pi \hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m \omega x^2}{2\hbar}\right) H_n\left(\frac{m \omega}{\hbar} x\right) $$

   $$ H_0(\xi) = 1, \quad H_1(\xi) = 2 \xi, \quad H_2(\xi) = 4 \xi^2 - 2, \quad \ldots $$

   Alternatively, if you chose to work in a representation based on number states $|n_x, n_y, n_z\rangle = |n_x\rangle |n_y\rangle |n_z\rangle$, you may use

   $$ z = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a_z + a_z^+) $$

   and similar expressions for $x$ and $y$. Also

   $$ \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} $$
Advanced Quantum Mechanics

Answer Two (2) out of Three (3) Questions.

25. (a) In the Born approximation, the scattering amplitude $f(\theta)$ is

$$f(\theta) = -\frac{m}{2\pi \hbar^2} \int V(r) \exp\left( i \frac{(p-q) \cdot r}{\hbar} \right) d^3r,$$

where $p$ is the incident momentum and $q$ is the scattered momentum and $V(r)$ is the scattering potential. Find $f(\theta)$ when $V(r) = g \delta(r)$ and $\delta(r)$ is the three-dimensional delta function.

(b) For short-ranged potentials, the total cross section is

$$\sigma = 4\pi b^2,$$

where $b$ is the scattering length. Relate $b$ to $g$. (Assume $b$ and $g$ have the same sign.)

(c) The atomic scattering of thermal neutrons of wavelength $\lambda$ can be accurately characterized by a scattering length. Assume neutrons of mass $m$ are incident from vacuum with energy $E$ on a material with atomic density $N$. Further assume each atom scatters with the same scattering length $b$. Hence, you may consider the potential energy $V_{av}$ in the material to be the potential $V(r)$ averaged over a large volume.

(i) Find the wavenumber $k$ for the neutrons inside the material in terms of $E$ and $V_{av}$.

(ii) Show using Snell's law, that the index of refraction of the material is $n = k/k_0$, where $k_0$ is the wavenumber for the neutrons in vacuum.

(iii) Assuming $|V_{av}| \ll E$, show that the material behaves as a medium with a refractive index

$$n = 1 - \alpha \lambda^2.$$

Find $\alpha$ in terms of $N$ and $b$. 

\[\boxed{31}\]