DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part I

Tuesday, August 28 2001
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Classical Physics

1. __________
2. __________
3. __________
4. __________

Student Identification Code: __________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2001.
Answer Three (3) out of Four (4) Questions

1. When a point particle reflects elastically from a smooth surface, the angle of incidence $\theta$ is equal to the angle of reflection $\theta'$ (see figure). This rule is modified when a sphere of radius $R$, mass $M$, and moment of inertia $I = \gamma MR^2$ scatters from a smooth but "sticky" surface. Find an expression for

$$\frac{\tan \theta'}{\tan \theta}$$

in terms of the moment of inertia coefficient $\gamma$ using the following assumptions.

(a) Before striking the sticky surface, the sphere is not rotating.

(b) For motion perpendicular to the surface, energy is conserved, so the vertical component of the speed is reversed (but unchanged in magnitude) by the collision.

(c) While in contact with the sticky surface the sphere is rolling. Hence, after striking the sticky surface, the sphere is rotating.
Answer Three (3) out of Four (4) Questions

2. A uniform frictionless chain of length $L$ and mass $\rho$ per unit length falls under the influence of gravity from the edge of a desk as shown in the Figure.

   (a) Let $x(t)$ be the length of the portion of the chain hanging down from the desk at time $t$. Obtain a differential equation for $x(t)$.
   Hint: Consider the time dependence of the momentum. *Note that the portion of the chain piled up at the edge of the desk is not being accelerated.*

   (b) Solve this differential equation for $x(t)$.
   Hint: Try motion with constant acceleration. Assume the chain starts to fall at time $t = 0$.

   (c) Find the time $T$ and the speed $v(T)$ when the entire chain leaves the desk.

   (d) Find the total mechanical energy (kinetic plus potential) at time $T$ when the entire chain leaves the desk. Is the energy conserved throughout the fall? Explain briefly.
Answer Three (3) out of Four (4) Questions

3. (a) Consider an infinitely long straight wire carrying current $I$. Use Ampere's law to find the magnetic field $\mathbf{B}(r)$ as a function of the radial distance $r$ from the wire. Without evaluating the integral, write down how you would have calculated $\mathbf{B}(r)$ using Biot-Savart's law.

(b) Now consider two infinitely long straight wire segments carrying current $I_1$ which connect at point $O$ with a 90° angle. What is the magnetic field $\mathbf{B}(r)$ (magnitude and direction) at a point $P$ located in the same plane as the wires a distance $s$ exactly below the 90° corner (see Figure 1 below). Hint: Evaluate Biot-Savart's law for each wire segment by making use of the structure of the integrand, the integral limits and your result from part (a). Do not use brute force integration.

(c) If a straight wire segment located in the same plane as the wires and carrying current $I_2$ in the direction indicated extends from a distance $a$ to a distance $b$ below the corner (see Figure 2 below), what is the force (magnitude and direction) on this wire segment?

![Figure 1](image1.png)

![Figure 2](image2.png)
Answer Three (3) out of Four (4) Questions

4. A very long coaxial cable of inner radius $a$ and outer radius $b$ is held at a constant potential difference $V_0$ at one end. At the other end of the cable is a resistor $R$ allowing a current $I = V_0/R$ to flow (see Figure). The cable itself has zero resistance.

(a) Solve Laplace’s equation in cylindrical coordinates with the appropriate boundary conditions for the potential $V(r)$ inside the coaxial cable. Find the electric field $E(r)$ from the potential.

Hint: Since there is no $\varphi$- or $z$-dependence $\nabla^2 V = \frac{1}{r \, dr} \left( r \, \frac{dV}{dr} \right)$, where $r$ is the radial coordinate.

(b) Using Ampere’s law find the magnetic field $B(r)$ for $r < a$, $a \leq r < b$, and $b \leq r$.

(c) Find the Poynting vector $S(r)$ inside the coaxial cable. In which direction does $S(r)$ point. Sketch $E$, $B$, and $S$.

(d) Show that the power transported down the cable obtained by integrating the Poynting vector over the coaxial cable cross section equals the Joule heat dissipated in the resistor $R$.
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part II

Tuesday, August 28, 2001
1:00 - 3:00 PM

Instruction:

Answer one from Questions 5 and 6 and one from Questions 7 and 8

Please check below the two (2) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Advanced Classical Physics

5. 

6. 

7. 

8. 

Student Identification Code: 

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2001.
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

5. Consider a model for an infinitely long linear chain of molecules with lattice constant $a$ as shown in the Figure below. All the molecules have the same mass $m$, and all the springs have the same length $a$ (the lattice constant) in equilibrium and the same spring constant $k$. Assume that the small displacement $q_j$ of the $j$th molecule from the equilibrium position coincides with the displacement $q_{j+N}$ of the $(j+N)$th molecule, that is, the displacement variables obey the periodic boundary condition: $q_j = q_{j+N}$.

(a) Find the equations of motion for small oscillations.

(b) Show that the boundary condition is satisfied by the solution of the following form,

$$q_j(t) = \sum_n C_n \exp \left[ i(\omega_n t - K_n d_j) \right],$$

where $K_n = 2\pi n/(aN)$, with $-N/2 < n \leq N/2$ ($N$ is an even number) and $d_j = ja$.

(c) Find the normal frequencies $\omega_n$.

(d) Discuss the relative motions of two neighboring molecules for the following two special modes:

(i) The mode with $n = N/2$ ($C_n = 0$ for $n \neq N/2$ and $C_{N/2} \neq 0$), and

(ii) the mode with $n = 0$ ($C_n = 0$ for $n \neq 0$ and $C_0 \neq 0$).
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

6. One way to derive equations of special relativity (in one dimension) is to start with the Hamiltonian

\[ H(p, x) = \sqrt{(pc)^2 + (mc^2)^2} + V(x) \]

where \( H \) corresponds to the energy, \( p \) is the momentum, \( x \) is the position, \( m \) is the mass, \( c \) is the speed of light, and \( V(x) \) is the potential energy. The dynamics are given by Hamilton's equations

\[ \frac{\partial x}{\partial H} = \frac{\partial}{\partial p}, \quad \frac{\partial p}{\partial H} = -\frac{\partial}{\partial x}. \]

(a) Expand the square root of the Hamiltonian to lowest order in \( p^2 \) to find an approximate Hamiltonian for small momentum.

(b) Use Hamilton's equation to derive a relativistic expression for the momentum in terms of the velocity, the mass and the speed of light.

(c) Find the time-dependent position of a particle for the following example:

\[ V(x) = -kx, \] where \( k \) is a positive constant, with the initial condition that at \( t = 0, x = p = 0 \).

An alternative way to derive equations of special relativity (in one dimension) is to start with the Lagrangian

\[ L = -mc \sqrt{c^2 - \left(\frac{dx}{dt}\right)^2} - V(x) \]

and the Lagrange equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial (dx/dt)} \right) - \frac{\partial L}{\partial x} = 0. \]

(d) Show that for the example of part (c) above, the Lagrangian and Hamiltonian approach give the same differential equation for the position \( x(t) \).
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

7. Maxwell's equations in free space are given by

\[ \nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \]

(a) Derive the continuity equation,

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{S} = 0, \]

where \( u \) is the energy density and \( \mathbf{S} \) is the Poynting vector.

Hint: \( \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \)

(b) Show that the following standing waves,

\[ \mathbf{E} = \hat{y} E_0 \sin(kx) \sin(\omega t) \]

and

\[ \mathbf{B} = \hat{z} B_0 \cos(kx) \cos(\omega t), \]

satisfy Maxwell's equations if \( E_0 \) is suitably related to \( B_0 \) and \( \omega \) is suitably related to \( k \). What are these relationships?

(c) For the standing wave, obtain the energy density as a function of \( x \) and \( t \).

(d) Find the Poynting vector and indicate its direction.

(e) Calculate the time average power flow across a unit area in the \( yz \) plane.
Answer one (1) from Questions 5 and 6 and one (1) from Questions 7 and 8.

8. A point charge $q$ moves along the $z$-axis with constant velocity $v$ as shown in the Figure below, passing through the origin at time $t = 0$.

(a) Using $\delta$ functions write down the charge and current densities corresponding to the moving charge.

(b) Find the potential $V(r, z)$ and the vector potential $A(r, z)$ of the moving charge in the quasistationary (no retardation effects) approximation.

(c) Find the electric field $E(r, z)$ and the magnetic field $B(r, z)$ of the moving charge. Show that $B$ can be written as $B = (1/c^2)v \times E$.

(d) Find the electric and magnetic force on a second charge $Q$ which is moving parallel to the $z$-axis at a distance $d$ directly above the first charge with the same velocity $v$ (see Figure). At what velocity does the net force become equal to zero?
DEPARTMENT OF PHYSICS
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Comprehensive Field Examination

Part III

Thursday, August 30, 2001
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Statistical and Thermal Physics

9. 

10. 

11. 

12. 

Student Identification Code: 

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2001.
Answer Three (3) out of Four (4) Questions.

9. The difference between the heat capacity at constant pressure, \( C_p \), and the heat capacity at constant volume, \( C_v \), is defined as

\[
C_p - C_v = T \left( \frac{\partial S}{\partial T} \right)_p - T \left( \frac{\partial S}{\partial T} \right)_V .
\]

(a) Show that

\[
C_p - C_v = T \left( \frac{\partial P}{\partial V} \right)_T \frac{1}{\frac{\partial V}{\partial T}} .
\]

(b) Express \( C_p - C_v \) in terms of the temperature, volume, bulk modulus \( B = -V \left( \frac{\partial P}{\partial V} \right)_T \) and the volume coefficient of thermal expansion \( \beta = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p \).

(c) Using the expressions from part (b), calculate the bulk modulus and the volume coefficient of thermal expansion for an ideal gas. Find \( C_p - C_v \) for one mole of an ideal gas using the equation from part (a). Does your answer agree with the expected result?

Hints: (1) If the entropy is expressed as a function of temperature and pressure:

\[
dS = \left( \frac{\partial S}{\partial T} \right)_p dT + \left( \frac{\partial S}{\partial P} \right)_T dP .
\]

(2) When a system is heated at constant volume, its pressure change is given by:

\[
dP = \left( \frac{\partial P}{\partial T} \right)_V dT
\]

(3) A useful Maxwell relation is obtained from the derivatives of the Gibbs free energy, given by \( dG = -SdT + VdP \).

(4) The chain rule for temperature, pressure and volume is \( \frac{\partial P}{\partial T} \frac{\partial V}{\partial P} \frac{\partial P}{\partial V} = -1 \).

(5) If \( Y \) is an invertible function of \( X \), so that \( X \) can be written as a function of \( Y \),

\[
\frac{dX}{dY} = \left( \frac{dY}{dX} \right)^{-1}.
\]
10. Consider two heat reservoirs, one at a higher temperature $T_H$ and one at a lower temperature $T_L$. Use the first and second law of thermodynamics to answer the questions below. Hint: Consider entropy changes in the two heat reservoirs.

   (a) Show that if the system is isolated heat can only flow from the high temperature to the low temperature reservoir.

   (b) Find the maximum amount of mechanical work $W$ that can be extracted when heat flows from the high temperature to the low temperature reservoir. In terms of the temperatures involved derive the thermodynamic efficiency $W/Q$, where $Q$ is the heat extracted from the high temperature reservoir.

   (c) In a heat pump mechanical work $W$ is use to move heat from the low temperature reservoir to the high temperature reservoir. In terms of the temperatures involved derive the coefficient of performance $Q/W$ where $Q$ is the heat rejected at the high temperature reservoir.
11. Consider a degenerate super-relativistic electron gas for which the average energy of an electron $\varepsilon$ is much greater than the rest energy so that $\varepsilon \approx c\beta$ ($c$ is the speed of light). In general, the number of electrons in a phase space volume $Vd^3p$ is given by

$$dN = \frac{V}{\pi^2\hbar^3} n_\varepsilon(T) p^2 dp,$$

with the distribution function

$$n_\varepsilon(T) = \frac{1}{e^{(\varepsilon-\mu)\beta} + 1},$$

where $\mu$ is the chemical potential and $\beta = (k_B T)^{-1}$, where $k_B$ is the Boltzmann constant.

(a) Express the distribution function graphically as a function of the energy $\varepsilon$ in the limit $T \to 0$.

(b) Calculate the total number of electrons, $N$, in allowed states for the low temperature limit $T = 0$ as a function of $\mu$. Determine the chemical potential $\mu$ as a function of $N$.

(c) Obtain the total energy of the electron gas, $E$, as a function of $N$.

(d) Calculate the pressure $P$ at $T = 0$ and show that $PV = E/3$. 
Answer Three (3) out of Four (4) Questions.

12. A model of a paramagnetic solid is an array of $N$ independent spin = 1 atoms with magnetic moment $\mu$. In a magnetic field $B$ each atom can be in any of three energy levels

$$\varepsilon_e = \mu B$$

$$\varepsilon_0 = 0$$

$$\varepsilon_- = -\mu B$$

For this system, find expressions for

(a) The partition function.
(b) The free energy.
(c) The internal energy.
(d) The magnetization.
(e) The zero-field susceptibility.
(f) The entropy.
(g) The average number of atoms in the state with $\varepsilon_0 = 0$.
(h) Sketch the heat capacity as a function of temperature (for a fixed magnetic field) in the range $0 < k_B T < 2\mu B$. 

DEPARTMENT OF PHYSICS
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State University of New York

Comprehensive Field Examination

Part IV

Thursday, August 30, 2001
1:00 - 3:00 PM

Instruction:

Answer any three out of six questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Modern Physics

13. __________
14. __________
15. __________
16. __________
17. __________
18. __________

Student Identification Code: __________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2001.
Modern Physics

Fall 2001

Answer three (3) out of six (6) Questions.

13. For each of the two detectors listed below

(a) describe the fundamental process(es) by which the detected particle creates an electrical signal.

(b) give some estimate of the energy resolution of the detector and the principal cause that limits this energy resolution.

(c) discuss what processes contribute to background counts in the detector. Are detector background counts likely to be a significant problem in the applications described below?

Detectors and applications:

A. A silicon surface barrier detector such as those used to measure 2 MeV He ions backscattered from a target.

B. A large BGO or NaI scintillation counter such as those used to measure 4.4 MeV gamma rays coming from the target in $^{15}$N nuclear reaction analysis for hydrogen.
Answer three (3) out of six (6) Questions.

14.  (a) Estimate the binding energy of the \( n = 1, 2, \) and 3 levels in calcium (\( Z = 20 \)) using the Bohr model of the atom (same as the hydrogenic ion). Give your answer in atomic units (a.u.) and electron Volts (eV). Hint: 1 a.u. = \( e^2/a_0 = 27.2 \) eV, where \( a_0 \) is the Bohr radius.

(b) Estimate the energy of the \( K \) x-ray transition in calcium using the Bohr model.

(c) Estimate the energy of the electrons emitted in the \( KLL \) and the \( KLM \) Auger transitions in calcium using the Bohr model.

(d) Draw an energy level diagram indicating the above three transitions. Qualitatively discuss why and how observations in an actual atom would differ from the values estimated with the Bohr model.
Modern Physics

Answer three (3) out of six (6) Questions.

15. The primitive translation vectors of the hexagonal space lattice may be taken as

\[ a = \left( \frac{\sqrt{3}a}{2} \right) \hat{x} + \left( \frac{a}{2} \right) \hat{y}, \quad b = -\left( \frac{\sqrt{3}a}{2} \right) \hat{x} + \left( \frac{a}{2} \right) \hat{y}, \quad c = c \hat{z}. \]

(a) Find the volume of the primitive cell.

(b) Find the primitive translations of the reciprocal lattice so that the lattice is its own reciprocal, but with a rotation of axes.

(c) Describe and sketch the first Brillouin zone of the hexagonal space lattice.
16. A Geiger counter is placed near a material which contains two different radioactive isotopes. The time dependence of the number of counts per second (on a semi-log scale) is shown in the Figure below. For low count rates, assume 1% of the radioactive decays of each isotope is counted. Obtain rough estimates of the following:

(a) The half life of each isotope.

(b) The number of nuclei for each isotope in the sample at \( t = 0 \).

(c) The dead time of the counter.
17. Pions \((m_{\pi}c^2 = 140 \text{ MeV})\) decay into muons \((m_{\mu}c^2 = 106 \text{ MeV})\) and massless neutrinos according to \(\pi^\pm \rightarrow \mu^\pm + \nu_{\mu}\), with a lifetime of \(\tau_\pi = 2.6 \times 10^{-8} \text{ s}\). The muons are also unstable, decaying with a somewhat longer lifetime of \(\tau_\mu = 2.2 \times 10^{-6} \text{ s}\) into electrons and neutrinos.

(a) Find the kinetic energy of the muon and the ratio of its velocity to the speed of light if the pion is \emph{at rest} immediately before the decay.

(b) How far will the muon from part (a) travel before it disintegrates.

(c) High energy cosmic ray muons are produced in the upper atmosphere (at 8000 m, say) and travel toward the earth at very nearly the speed of light (0.998 \(c\), say). What is the \emph{sequence} of events by which these cosmic ray muons are produced? How far will these high energy muons go before disintegrating? Would they be able to make it to ground level?

(d) Now analyze the same process from the perspective of the muon. (In it own reference frame it only lasts \(2.2 \times 10^{-6} \text{ s}\). Given that the speed of light is \(c = 3 \times 10^8 \text{ m/s}\), does it make it to ground?

Comment: Use a single value for the lifetime of the muon, not an exponential decay law.
Answer three (3) out of six (6) Questions.

18. The lowest order Feynman diagrams corresponding to some elementary electromagnetic interaction processes are given below:

(a) Indicate which diagram corresponds to each of the following processes [For example, (5) = (A)]:

(A) Pair creation, (B) Pair annihilation, (C) Compton effect, (D) Bremsstrahlung, (E) Photoelectric effect, (F) Vacuum polarization, (G) Electron self-energy or mass renormalization, (H) Electron-positron scattering.

(b) What is pair creation?

(c) Pair creation as shown in Figure (5) cannot occur in empty space. (That is a Feynman diagram with one vertex only does not represent a possible physical process.) Explain why a photon alone cannot create the pair. Draw the lowest order diagram for pair creation which gives a non-zero result.

Note: Particles Antiparticles Photons
DEPARTMENT OF PHYSICS

University at Albany
State University of New York

Comprehensive Field Examination

Part V

Saturday, September 1, 2001
9:00 - 11:00 AM

Instruction:

Answer any three out of four questions

Please check below the three (3) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Quantum Mechanics

19. __________
20. __________
21. __________
22. __________

Student Identification Code: __________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2001.
Quantum Mechanics

Fall 2001

Answer 3 out of 4 problems

19. (a) Write down the Schroedinger equation for a spherically symmetric, three-dimensional harmonic oscillator of mass $m$ and frequency $\omega$.

(b) Using an unnormalized trial wave function of the form $\psi = e^{-\beta r}$, where $\beta$ is a variable parameter, use the variational method to obtain an upper limit for the energy of the ground state of the three-dimensional harmonic oscillator in units of $\hbar \omega$.

(c) Compare your result with the expected exact value and comment on why it differs.

Hints:

$$\int_{0}^{\infty} x^n e^{-ax} dx = n! / a^{n+1}$$

For a spherically symmetric wave function

$$\nabla^2 \rightarrow \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}$$
20. Consider a particle with mass $m$ and charge $e$ in a two dimensional box with sides along the $x$- and $y$-directions, each of length $a$ and with the center of the box at the origin.

(a) Write down the energy eigenvalues for the ground and first excited energy levels and the corresponding eigenfunctions. What are the orders of degeneracies for these two levels?

(b) Using the eigenfunctions for these levels find, without performing a detailed calculation, the matrix elements of the dipole moment operator $\mu_x = ex$ in the $x$-direction which are zero.

(c) Using the eigenfunctions for these levels, determine the non-zero matrix elements of the dipole moment operator $\mu_x$. Display all the zero and non-zero matrix elements in a matrix.

(d) Determine the corresponding matrix for the $xy$-component $Q_{xy} = exy$ of the quadrupole moment tensor. Display all the zero and non-zero matrix elements in a matrix.

Hints:

$$\int_{-\pi/2}^{\pi/2} u \sinudu = 2, \quad \int_{-\pi/2}^{\pi/2} u \sin^3 u du = \frac{14}{9}.$$
21. The Schrödinger equation for the spatial part of the wave function describing the two-electron helium atom is given by the expression

\[
\left[ -\frac{\hbar^2}{2m} \nabla_r^2 - \frac{\hbar^2}{2m} \nabla_{r_2}^2 - \frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}} \right] \psi(r_1, r_2) = E \psi(r_1, r_2),
\]

where \( r_1, r_2 \) are the coordinates of the two electrons with respect to the nucleus and \( r_{12} \) is the distance between the two electrons.

(a) Given that the Hamiltonian is unchanged when the coordinates of the two electrons are interchanged, how does the wave function behave when the spatial coordinates of the two electrons are interchanged? That is, what is the relation between \( \psi(r_1, r_2) \) and \( \psi(r_2, r_1) \)?

(b) Show that in a two electron system the angular momentum operators \( \hat{L}_z \) and \( \hat{L}_{2z} \) of the two electrons, respectively, do not individually commute with the electron-electron interaction \( e^2/r_{12} \), but that their sum \( \hat{L}_z = \hat{L}_z + \hat{L}_{2z} \) does commute.

(c) Show that \( \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \) commutes with \( e^2/r_{12} \). What are the possible values of the quantum number \( L \) in terms of the corresponding values \( l_1 \) and \( l_2 \)?

Hints: Since all three coordinate directions are equivalent you need to demonstrate the result for one angular momentum component only, e.g. the z-component.

You may evaluate the z-component of the angular momentum operators either in cartesian coordinates, or, if you are familiar with the ‘addition theorem’ of the spherical harmonics, in spherical coordinates by making use of

\[
\frac{1}{r_{12}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4 \pi}{2l + 1} \frac{r_s^l}{r_{12}^{l+1}} Y^*_l m (\theta_1, \phi_1) Y_l m (\theta_2, \phi_2).
\]
Quantum Mechanics

Fall 2001

Answer 3 out of 4 problems

22. A single electron is placed in a isotropic two-dimensional harmonic oscillator potential.

(a) Write down the energy in terms of the number of oscillator quanta in the two directions and show that the energy levels are:

One level with energy \( \varepsilon_1 = \hbar \omega \).
Two degenerate levels with energy \( \varepsilon_2 = 2\hbar \omega \).
Three degenerate levels with energy \( \varepsilon_3 = 3\hbar \omega \).

(b) Six electrons are placed in this potential well. Ignore electron-electron interactions, but do not ignore the Pauli exclusion principle or the spin of the electrons. Find the total ground state energy \( E \) (= the sum of the energies of the occupied single-particle levels) of this six-electron system for the following cases:

(i) There is no constraint on the total spin of the system.
(ii) The total spin is \( S = 1 \) (in units of \( \hbar \)).
(iii) The total spin is \( S = 2 \).
(iv) The total spin is \( S = 3 \).
DEPARTMENT OF PHYSICS
University at Albany
State University of New York

Comprehensive Field Examination

Part VI

Saturday, September 1, 2001
1:00 - 3:00 PM

Instruction:

Answer any two out of three questions

Please check below the two (2) problems you have done and return this sheet with your examination books. Each problem MUST be done in a separate examination book. The problems within each area carry equal weight.

Advanced Quantum Mechanics

23. ________
24. ________
25. ________

Student Identification Code: ________

NOTE: This same Code is to be used on all sections of the Comprehensive Examination taken in Fall 2001.
Answer two (2) out of three (3) Questions.

23. A hydrogenic atom (nuclear charge +Ze) in its ground state is subjected to a perturbation potential of the form

\[ H' = (-e)(-Fz) = (er\cos\theta)F = (4\pi/3)^{1/2}e\rho Y_{1,0}(\theta,\phi)F, \]

where \( \theta \) is the angle between the z-direction and the radius vector of the electron with respect to the nucleus, \( F \) being a uniform electric field applied to the atom in the z-direction. \( Y_{1,0}(\theta,\phi) \) is a spherical harmonic.

(a) Using perturbation theory and considering all the excited states corresponding to \( n = 2 \) obtain the first order perturbed wave function for the ground state. What is the first order change in energy for the ground state?

(b) Find the second order change in energy, \( \delta E^{(2)} \), for the ground state and from it the dipole polarizability \( \alpha_d \) using the relation

\[ \delta E^{(2)} = -(1/2) \alpha_d F^2, \]

again considering all the excited states corresponding to \( n = 2 \).

(c) Using the perturbed wave function from part (a), obtain the expectation values for the x-, y-, and z-components of the electric field produced by the perturbed electron at the site of the nucleus (the coordinate origin). Comment on your result in one or two sentences.

Hints: For a hydrogenic atom with nuclear charge +Ze

\[ \psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi), \quad E_{nlm} = E_n = -\frac{Z^2}{2n^2} \frac{e^2}{a_0}, \quad a_0 \text{ is the Bohr radius.} \]

\[ Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{1,\pm1}(\theta, \phi) = \pm \sqrt{\frac{3}{8\pi}} e^{\pm i\phi} \]

\[ R_{1,0}(r) = 2 \left( \frac{Z}{a_0} \right)^{3/2} \exp \left( -\frac{Zr}{a_0} \right), \quad R_{2,0}(r) = 2 \left( \frac{Z}{2a_0} \right)^{3/2} \left( 1 - \frac{Zr}{2a_0} \right) \exp \left( -\frac{Zr}{2a_0} \right), \]

\[ R_{2,1}(r) = \frac{1}{\sqrt{3}} \left( \frac{Z}{2a_0} \right)^{3/2} \frac{Zr}{a_0} \exp \left( -\frac{Zr}{2a_0} \right), \]

\[ \int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}} \]
Answer two (2) out of three (3) Questions.

24. A free electron is subjected to a suddenly created Coulomb potential \( V(r) = -\frac{1}{r} \).

This sudden potential could appear through neutron decay. Let the probability that the electron is captured into the hydrogen atom 1s ground state by this potential be \( P(p) \), where \( p = m_0v \) is the momentum of the free electron with \( m_0 \) the electron mass.

(a) Deduce the momentum-dependence of this capture probability by calculating the ratio

\[
R = \frac{P(p)}{P(p = 0)}.
\]

To calculate this ratio, assume that the electron is moving along the z-direction. Describe qualitatively the nature of the dependence of \( R \) on the magnitude of \( p \) and comment briefly on the physical reason for this dependence.

(b) The electron may also be captured into excited states with non-zero orbital angular momentum. If the electron is traveling along the z-direction, which 3d state(s) will have nonzero capture probabilities: the state with \( m = 2, m = 1, m = 0, m = -1, m = -2 \)? Explain briefly.

(c) Assume the proton which produces this suddenly appearing Coulomb potential is also moving along the z-axis with a velocity \( v = p/\hbar \). Modify your formula of part (a) to account for the proton motion. Assume that the proton mass \( M \gg m_0 \). What is the condition on the proton velocity which will maximize the capture probability into the 1s ground state? Assume the two velocities are non-relativistic.

NOTE: The hydrogen atom ground state wave function is (\( a_0 \) is the Bohr radius)

\[
\psi_{1s}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_0^3}} \exp\left(-\frac{r}{a_0}\right).
\]

If \( k = p/\hbar \) is along the z-axis, \( k \cdot r = kr \cos \theta \) and for the angular integration in spherical coordinates \( \sin \theta \, d\theta = -d(\cos \theta) \). Also \( \int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}} \), where \( a \) may be a complex constant.

You may also want to use

\[
\exp(ik \cdot r) = \sum_{l=0}^\infty (2l + 1) i^l j_l(kr) P_l(\cos \theta),
\]

with \( j_0(kr) = \frac{\sin kr}{kr}, j_1(kr) = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}, j_2(kr) = \left( \frac{3}{(kr)^3} - \frac{1}{kr} \right) \sin kr - \frac{3}{(kr)^2} \cos kr \).
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Answer two (2) out of three (3) Questions.

25. In second quantization the Hamiltonian of the isotropic three-dimensional harmonic oscillator may be written as

$$\hat{H} = \hbar \omega (\alpha_x^* \alpha_x + \alpha_y^* \alpha_y + \alpha_z^* \alpha_z + \frac{1}{2})$$

with $\alpha_x |n_x\rangle = \sqrt{n_x} |n_x - 1\rangle$ and $\alpha_x^* |n_x\rangle = \sqrt{n_x + 1} |n_x + 1\rangle$ and similar expressions for the $y$- and $z$-coordinate. $|n_x\rangle$ represents an eigenfunction with $n_x$ oscillator quanta in the $x$-direction.

(a) Write down the energy eigenvalue in terms of $n_x$, $n_y$, and $n_z$ and give the corresponding eigenfunctions. What is the degeneracy of the eigenstate for $n = 0$, 1, and 2 where $n = n_x + n_y + n_z$?

(b) Write down complete sets of basis states for the ground state ($n = 0$), the first excited state ($n = 1$), and the second excited state ($n = 2$).

(c) Express the angular momentum operator $l_z$ in terms of $\alpha_i$ and $\alpha_i^*$, where the subscript $i$ stands for $x$, $y$ and $z$, and show that the basis states from part (b) are not eigenfunctions of $l_z$.

Hint:

$$\hat{x} = \left( \frac{\hbar}{2m \omega} \right)^{1/2} (\alpha_x + \alpha_x^*)$$

$$\hat{p}_x = i \left( \frac{m \omega \hbar}{2} \right)^{1/2} (\alpha_x^* - \alpha_x)$$

(d) Find the eigenvalues and eigenfunctions of $l_z$ for the first excited state ($n = 1$).

(e) Find the angular momentum operator $\hat{l}^2$ in terms of $\alpha_i$ and $\alpha_i^*$ and determine its eigenvalues for the ground ($n = 0$), first ($n = 1$) and the second excited state ($n = 2$).

Hint: Start with $l_z^2$ and use cyclical permutations to obtain the contributions from all three components of the angular momentum operator. Note that some products of the operators $\alpha_i$ and $\alpha_i^*$ are diagonal within the set of basis states from part (b).

(f) From your result for part (e), what values of $l$ appear to be associated with a given value of $n$? What is the parity of the eigenstates of $\hat{l}^2$ for $n = 0$, 1, and 2? Compare with the corresponding relation between $n$ and $l$ for the hydrogen atom.