1] For each $n \in \mathbb{Z}$ define $B_n = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n - 1, n, n + 1\} & \text{if } n \text{ is even.} \end{cases}$

- Show that $\mathcal{B} = \{B_n | n \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$.
- Show that, with respect to this topology, $\{n\}$ is open if $n$ is odd, and $\{n\}$ is closed if $n$ is even.
- Show that, with respect to this topology, $\{n \mid n \text{ is odd}\}$ is dense in $\mathbb{Z}$.

2] Let $X$ be a metric space with metric $d$, and fix a point $x_0 \in X$.
- Show that the function $d(x_0, -): X \to \mathbb{R}, \ x \mapsto d(x_0, x)$ is continuous.
- Let $\mathcal{T}$ be a topology on $X$ such that the function $d(x_0, -)$ is continuous with respect to $\mathcal{T}$. Show that $\mathcal{T}$ is finer than the metric topology.

3] Let $X$ and $Y$ be topological spaces, and let $f, g: X \to Y$ be continuous maps. Prove that, if $Y$ is Hausdorff, then $\{x \in X \mid f(x) \neq g(x)\}$ is open in $X$.

4] Prove the tube lemma: Let $X$ and $Y$ be topological spaces, with $X$ compact. Let $y_0 \in Y$ and let $U$ be an open subset of the product $X \times Y$ such that $X \times \{y_0\} \subseteq U$. Prove that there exists an open set $V \subseteq Y$ such that $X \times \{y_0\} \subseteq X \times V \subseteq U$.

5] Let $X$ be a second-countable topological space, and let $S \subseteq X$. Prove that, if $S$ is uncountable, then uncountably many points of $S$ are limit points of $S$.

6] The suspension $\Sigma X$ of a topological space $X$ is defined as the quotient space

$$\Sigma X = \frac{X \times [0,1]}{\sim}$$

where $(x,t) \sim (y,s)$ if and only if $s = t = 0$, or $s = t = 1$, or $(x,t) = (y,s)$.

- Show that $\Sigma S^1$ is homeomorphic to $S^2$.
- Is $\Sigma X$ connected for every space $X$? Prove it or construct a counterexample.
- Is $\Sigma X$ simply connected for every space $X$? Prove it or construct a counterexample.

7] Recall that the real projective plane $\mathbb{R}P^2$ is the space obtained from $S^2$ by identifying antipodal points.

- Show that the quotient map $S^2 \to \mathbb{R}P^2$ is a covering map.
- Describe three pairwise non-homeomorphic two-fold coverings of $\mathbb{R}P^2 \vee S^1$.

8] Let $p: E \to S^1 \vee S^1$ be a covering map.

- Let $\alpha, \beta: [0,1] \to E$ be paths such that $\alpha(0) = \beta(0)$. Assume that $p \circ \alpha$ and $p \circ \beta$ are path homotopic. What can you conclude about $\alpha$ and $\beta$? Explain which results you are using.
- Using only your answer to the previous question and an appropriately chosen covering, show that $\pi_1(S^1 \vee S^1, b)$ is not abelian, where $b \in S^1 \vee S^1$ is the point of intersection of the two circles.