1) Suppose \( p : E \to B \) is a covering map and \( f, g : [0,1] \to B \) are continuous maps such that \( pf = pg \) and \( f(0) = g(0) \). Prove that \( f(t) = g(t) \) for all \( t \) in \([0,1]\).

2) Let \( X \) be a 2\(^{nd}\) countable space.
   a) Prove that at most countably many points of \( X \) can be isolated points.
   b) Prove that every open cover of \( X \) has a countable subcover.

3) a) Let \( r : X \to A \) be a retraction and let \( i : A \to X \) be the inclusion map and let \( r^* \) and \( i^* \) be the induced homomorphisms of fundamental groups with some common base point. Assuming functorial properties of induced maps, what can you say about \( r^* \) and \( i^* \)?

   Now using part a as needed, tell whether each of the following is true or false and justify your answer. You can also assume the fundamental groups of specific spaces are known.
   b) If \( A \) is a retract of \( X \) and \( A \) is simply connected, so is \( X \).
   c) If \( A \) is a retract of \( X \) and \( X \) is simply connected, then so is \( A \).
   d) There is no retract of the projective plane \( \mathbb{P}^2 \) onto a subspace which is homeomorphic to the circle \( S^1 \).
   e) There is a retraction of the closed unit disk in \( \mathbb{R}^2 \) onto its boundary.

4) Let \( I_P^\omega \) and \( I_B^\omega \) denote the countable product of unit intervals with the product and box topologies, respectively.
   a) Determine with proof whether the identity map from \( I_P^\omega \) to \( I_B^\omega \) is continuous or not.
   b) Same question for the identity map from \( I_B^\omega \) to \( I_P^\omega \).
   c) Let \( A \) denote the set of points which are zero in all but finitely many coordinates. Determine, with proof, the closure of \( A \) in each topology.
5) Recall that a surjective map $f : X \to Y$ is a quotient map provided that a set $U$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$.

a) Suppose $p : X \to Y$ and $q : Y \to X$ are continuous maps and $p \circ q = \text{the identity map of } Y$. Prove that $p$ is a quotient map.

b) Let $f : R \times R \to R$ be projection onto the first coordinate and let $p$ be the restriction of $f$ to the space $X$ consisting of all points $(x, y)$ with $x \geq 0$ or $y = 0$ (or both). Prove that $p$ is a quotient map that is neither an open map nor a closed map.

6) Let $X = P^2 \cup S^1$ denote the one point union of the projective plane and a circle, and let $Y$ denote the subset of $R^3$ that is the union of the sets $A$, $B$, $C$ where

\[
A = \{(x, y, z)|x^2 + y^2 \leq 1 \text{ and } z = 0 \text{ or } 1\}
\]

\[
B = \{(-\frac{1}{2}, 0, z)|0 \leq z \leq 1\}
\]

\[
C = \{(\frac{1}{2}, 0, z)|0 \leq z \leq 1\}
\]

a) Find the fundamental group of $X$ and justify your answer.

b) Find the fundamental group of $Y$ and justify your answer.

c) Pick one of these two spaces and describe its universal covering space.

7) Suppose $X$ and $Y$ are arbitrary spaces with $Y$ compact. Let $x_0$ be a point of $X$ and let $U$ be an open set in the product space $X \times Y$ that contains $x_0 \times Y$. Prove that there exists an neighborhood $V$ of $x_0$ such that $V \times Y$ is contained in $U$. 