Do as many problems as you can.

1. Let \( X \) be a compact space, and suppose there is a finite family of continuous functions \( f_i : X \to \mathbb{R}, i = 1, \ldots, n \), with the following separation property: Given \( x \neq y \) in \( X \) there is an \( i \) such that \( f_i(x) \neq f_i(y) \). Prove that \( X \) is homeomorphic to a subset of \( \mathbb{R}^n \).

2. Let \( f : X \to Y \) be a continuous function, and let \( G \subset X \times Y \) be its graph, that is, the subset \( G = \{ (x, y) \mid y = f(x) \} \). If \( Y \) is Hausdorff, prove that \( G \) is closed in \( X \times Y \).

3. (a) State Urysohn’s lemma and Tietze’s extension theorem.
   (b) Let \( X \) be a metric space and let \( \{x_n\}, n \geq 1 \), be an infinite sequence in \( X \) such that \( d(x_i, x_j) \geq 1 \) for \( i \neq j \). Prove that there is a continuous map \( f : X \to \mathbb{R} \) such that \( f(x_n) = n^2 \) for all \( n \).

4. Let \( G \) be the wedge of infinitely many unit intervals indexed by positive integers. That is, the set \( G \) is the union of \([0, 1], i = 1, 2, 3, \ldots \), with all 0-endpoints identified. Recall that the weak topology on \( G \) is the smallest collection of subsets such that the intersection with each closed edge is open within that edge. Prove that \( G \) with the weak topology is not metrizable.

5. The connected components of a topological space \( X \) are the equivalence classes for the following relation: \( x \sim y \) if and only if there is a connected subset of \( X \) containing both \( x \) and \( y \).
   (a) Prove this relation \( \sim \) is indeed an equivalence relation.
   (b) Prove that every connected component is connected.

6. Let \( \alpha \) and \( \beta \) be paths from \( x_0 \) to \( x_1 \) in a space \( X \). If \( \pi_1(X, x_0) = 0 \), prove that \( \alpha \simeq_p \beta \).

7. Let \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a covering. Suppose the following condition holds: For every nontrivial element \( [\alpha] \in \pi_1(X, x_0) \) and every representative loop \( \alpha \), the lift \( \tilde{\alpha} \) starting at \( \tilde{x}_0 \) is not a loop. Prove that \( \pi_1(\tilde{X}, \tilde{x}_0) = 0 \).

8. (a) Compute the fundamental group of \( S^1 \lor S^2 \lor S^3 \), the one-point union of the circle, the two-dimensional sphere, and the three-dimensional sphere.
   (b) Find the universal cover of \( S^1 \lor S^2 \lor S^3 \).