1. Let $S$ be a (Lebesgue) measurable subset of $\mathbb{R}$ and let $f, g : S \to \mathbb{R}$ be measurable functions. Prove that

(a) $f + g$ is measurable.

(b) If $\phi \in C(\mathbb{R})$, then $\phi(f)$ is measurable.

2. State and prove the Lebesgue (dominated) convergence theorem.

3. Use examples to illustrate that for sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$, uniform convergence, pointwise convergence and convergence in measure are different concepts.

4. Evaluate the following double integral. (Need to justify all steps!)

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx \, dy.$$ 

5. Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded Lebesgue measurable function. Define sequence $f_n : [0, 1] \to \mathbb{R}$ by $f_0(x) = 1$ and

$$f_{n+1}(x) = 1 + \int_0^x F(f_n(t)) dt, \quad n \geq 1.$$ 

(a) Show $f_n \in C[0, 1]$.

(b) Prove that there exists $f \in C[0, 1]$ and a subsequence $f_{n_k} \to f$ in $C[0, 1]$ (with the supreme norm $\| \cdot \|_\infty$).

6. Suppose $1 \leq p < \infty$ and $f_n, f \in L^p(0, 1)$. If $f_n \to f$ almost everywhere, then prove that $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

7. Describe how the Lebesgue measure on $\mathbb{R}$ is defined.

8. Give an example of a bounded positive function on $[0, 1]$ that is not Lebesgue integrable.