1. Let \( \epsilon \) be any fixed positive number. Construct an open subset \( U \subset \mathbb{R} \) such that \( \mathbb{Q} \subset U \) and the Lebesgue measure \( m(U) < \epsilon \).

2. Construct a subset in \( \mathbb{R} \) that is not Lebesgue measurable.

3. State and prove Lusin’s theorem.

4. Find a sequence of Lebesgue measurable functions \( \{f_n\} \) on \([0, 1]\) such that
   (a) \( \{f_n\} \) is convergent in measure.
   (b) \( \{f_n(x)\} \) is NOT convergent for any \( x \in [0, 1] \).

5. Show that the function \( f(x) = \ln(1/x) \) is in \( L^p(0, 1) \) for all \( 1 \leq p < \infty \).

6. Let \( f \in L^1(\mathbb{R}) \). Its Fourier transform is defined
   \[
   \hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}dx.
   \]
   Show that \( \hat{f} \) is bounded and continuous.

7. Let \( x \) and \( y \) be two vectors in a Hilbert space \( H \). Prove that \( \|x + cy\| \geq \|x\| \) for every complex number \( c \) if and only if \( x \) and \( y \) are orthogonal.

8. The Hardy space \( H^2(T) \) over the unit circle \( T \) is the Hilbert space with orthonormal basis \( \{z^k : k \geq 0\} \), where \( |z| \leq 1 \). For a fixed \( \lambda \in \mathbb{D} \) consider the linear functional \( F \) defined by \( F(f) = f(\lambda) \), \( \forall f \in H^2(T) \). Show that \( F \) is bounded. And find the function \( \phi \in H^2(T) \) such that
   \[
   F(f) = \langle f, \phi \rangle, \ \forall f \in H^2(T).
   \]