Problem 1. State the following

a. Egorov’s Theorem.
b. Fatou’s Lemma.
c. Monotone Convergence Theorem.
d. Lebesgue Dominated Convergence Theorem.
e. Hölders Inequality.

Problem 2. Prove (b) and (d).
PART II

Problem 3. Prove that the product of two measurable functions is measurable.

Problem 4. Give an example of a sequence of continuous functions $f_n$ with domain $[0, 1]$ converging pointwise to a continuous function $f$ also with domain $[0, 1]$ such that

$$\lim_{n \to \infty} \int f_n \, dx \neq \int f \, dx.$$ 

Problem 5. Show that if $f \in L^p(\mathbb{R})$, then

$$\int |f(x)|^p \, d\lambda(x) = \int_0^\infty p t^{p-1} \lambda\{x : |f(x)| > t\} \, dt,$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. Justify all your steps.

Hint: one way to prove this is to write the right hand side as a double integral.

Problem 6. Let $f$ be an integrable function on $\mathbb{R}$. Show that given any $\epsilon > 0$, there exists a simple function $g$ on $\mathbb{R}$ such that

$$\int |f(x) - g(x)| \, dx < \epsilon.$$ 

Problem 7. Let $\lambda$ be the Lebesgue measure on $[0, 1]$. Recall that if $f$ is a function with domain $[0, 1]$, then

$$\|f\|_\infty = \operatorname{ess \ sup}_{x \in [0,1]} |f(x)| = \inf\{M : \lambda(\{x \in [0,1] : |f(x)| > M\}) = 0\},$$

and if $0 < p < \infty$,

$$\|f\|_p = (\int_0^1 |f(x)|^p \, dx)^{1/p}.$$

Prove that if $f$ is a measurable function with domain $[0, 1]$ and $\|f\|_\infty < \infty$, then

a. $f \in L^p([0, 1])$ for all $p > 0$ and $\|f\|_p \leq \|f\|_\infty$,

b. if $1 \leq p < q < \infty$ then $\|f\|_p \leq \|f\|_q$,

c. $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$. 

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