PART I: Do all of the following problems.

1. Show that the interval $[0, 1]$ is uncountable.

2. Prove that the set of continuous functions on $[0, 1]$ (denoted by $C[0, 1]$) with uniform metric is a complete metric space.

3. State the following theorems:
   
   a, Fatou’s Lemma;
   b, Lebesgue Dominated Convergence Theorem;
   c, Baire’s Category Theorem;
   d, Radon-Nikodym Theorem.

4. Prove 3(b).
PART II: Do at least four of the following problems.

5. Prove 3(c).

6. Let $\lambda$ be the Lebesgue measure on $[0, 1]$, and let $f$ be defined on $[0, 1]$ such that $f(x) = 0$ if $x$ is irrational and $f(x) = 1/n$ if $x = m/n$ where $m$ and $n$ are coprime nonnegative integers. Show that $f$ is Riemann integrable.

7. Let $f$ be an integrable function on $[0, 1]$. Show that if $\int_0^c f(x)dx = 0$ for every $c \in [0, 1]$, then $f = 0$ almost everywhere on $[0, 1]$.

8. Let $\{f_n\}$ be a sequence in $L^2(0, 1)$ that converges to $f$ in norm (i.e. $\lim_{n \to \infty} \|f_n - f\| = 0$). Show that $f_n$ converges in measure to $f$.

9. Let $\mu$ be a Borel measure on $[1, 2]$ such that $\mu(a, b) = \ln(b) - \ln(a)$ for every $a, b \in [1, 2]$. Show that $\mu << \lambda$, where $\lambda$ is the Lebesgue measure, and find $d\mu/d\lambda$.

10. Let $f$ be in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(a) Show that $f \in L^p(\mathbb{R})$ for every $1 \leq p \leq 2$.

(b) Show that

$$\lim_{p \to 1^+} \|f\|_p = \|f\|_1.$$