University at Albany
Department of Mathematics and Statistics
Probability Preliminary Examination
January 2010

Do all 4 problems.

Problem 1. Let \( \{X_n\} \) be independent identically distributed random variables with mean 0 and variance 1. Let \( S_n = \sum_{i=1}^{n} X_i \) and let \( \tau \) be a stopping time with respect to filtration given by \( \{X_n\} \), that is \( \{\tau = j\} \) is measurable with respect to \( \sigma(X_1, X_2, \ldots, X_j) \), such that \( E(\tau) < \infty \).

(a) Explain why \( X_n \) and \( 1_{\{\tau \geq n\}} \) are independent.
(b) Show \( E(S_\tau^2) = E(S_{\tau \wedge (n-1)}^2) + P(\tau \geq n) \).
(c) Use that \( S_{\tau \wedge (n+k)} = S_{\tau \wedge n} + \sum_{i=n+1}^{n+k} X_i 1_{\{\tau \geq i\}} \) to show that \( \{S_{\tau \wedge n}\} \) is a Cauchy sequence in \( L^1 \). (Note: The \( L^1 \)-norm of a random variable \( Y \) is simply \( E(|Y|) \).)
(d) Use the above to show \( E(S_\tau^2) = E(\tau) \)

Problem 2.

(a) Prove that if \( \sum_{n=1}^{\infty} P(A_n) < \infty \) then \( P(A_n \text{ i.o.}) = 0 \).
(b) Prove that if \( \sum_{n=1}^{\infty} P(A_n) = \infty \) and \( \{A_n\} \) are independent events then \( P(A_n \text{ i.o.}) = 1 \).
(c) Construct a sequence \( \{X_n\} \) taking values \( \{0, 1\} \) such that \( X_n \to 0 \) in probability but \( \lim \sup_{n \to \infty} X_n = 1 \) with probability one.

Problem 3. Let \( \{A_n\} \) be a sequence of independent events such that \( \sum_{n=1}^{\infty} P(A_n) = \infty \). Show that

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} 1_{A_i}}{\sum_{i=1}^{n} P(A_i)} = 1 \quad \text{in probability.}
\]

Problem 4. For \( \epsilon \in (0, 1) \), let \( \{X_n^\epsilon\} \) be i.i.d. random variables with \( P(X_n^\epsilon = \epsilon) = P(X_n^\epsilon = -\epsilon) = 1/2 \). Let \( N^\epsilon \) be a Poisson random variable with parameter \( \epsilon^2/2 \), independent of \( \{X_n^\epsilon\} \). Let

\[
Y_\epsilon = \sum_{i=1}^{N^\epsilon} X_i^\epsilon.
\]

(a) Compute the characteristic function of \( Y_\epsilon \), \( E(e^{itY_\epsilon}) \).
(b) Find \( \lim_{\epsilon \to 0} E(e^{itY_\epsilon}) \). What does this say about the convergence in distribution of \( Y_\epsilon \) as \( \epsilon \to 0 \)? Explain.