Department of Mathematics and Statistics
Preliminary Examination in Probability
September 2006

Do all 6 problems.

1. Suppose a machine flips a fair coin indefinitely and at each flip it records whether it came up head or tail. We may start looking at the results at the $n^{th}$ flip of the coin and we wonder how many heads in a row we will see. Let $l_n$ be the number of consecutive heads we see starting from the $n^{th}$ toss, that is $l_n = k$ if the tosses numbered $n, n+1, \ldots, n+k-1$ are all heads and the $n+k$ toss is a tail.
   (a) Find $P(l_n \geq n$ for infinitely many $n$’s)
   (b) Find $P(l_n = 1$ for infinitely many $n$’s)

2. Let $X_1, X_2, \ldots, X_n \ldots$ be iid.
   (a) Prove $E|X_1| < \infty \iff P(|X_n| > n \text{ for infinitely many } n \text{’s}) = 0$
   (b) State the Strong Law of Large Numbers.
   (c) Explain why the Strong Law of Large Numbers fails when $E|X_1| = \infty$

3. Show that if $Y_n \rightarrow Y$ in distribution and $X_n \rightarrow c$ in probability where $c > 0$ is a constant, then $Y_n/X_n \rightarrow Y/c$ in distribution.

4. Let $X_1, X_2, \ldots$ be iid with $E(X_1) = 0$ and $E(|X_1|^2) = 4$. Find the limit in distribution of
   $$\sqrt{n} \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^2}$$

5. Let $Y_1, Y_2, \ldots, Y_n, \ldots$ be independent Poisson random variables with parameter $\lambda$.
   (a) Find the characteristic function of $Y_n$
   (b) Find the characteristic functions of $S_n = Y_1 + Y_2 + \ldots + Y_n$
   (c) Find the characteristic function of $(S_n - n\lambda) / \sqrt{n\lambda}$ and its limit (with justification) as $n \rightarrow \infty$.
   (d) What does the limit in part (c) tell you about the limiting distribution of $(S_n - n\lambda) / \sqrt{n\lambda}$?

6. Consider the interval $[0, 1]$. Let $\beta_n$ the $\sigma$-algebra generated by the dyadic intervals of length $2^{-n}$, that is $[0, 1/2^n], [1/2^n, 2/2^n], [2/2^n, 3/2^n], \ldots, [2^{n-1}/2^n, 1]$.
   Let $f$ be an integrable function on $[0, 1]$.
   (a) Give an expression for $E(f|\beta_n)$. (Try $n = 1, n = 2$ then derive the general expression)
   (b) Show that $E(f|\beta_n)$ is a martingale.
   (c) Find $\lim_{n \rightarrow \infty} E(f|\beta_n)$