Work all 7 problems. You have 3 hours. The prelim is closed book.

1. Let $Z_1, Z_2, \ldots, Z_{15}$ be 15 independent normal random variables with mean 0 and variance 1. Describe the distribution of the following random variables. (Your answers might be something like: normal with mean 0 and variance 5, or 3 times a chisquare with 7 degrees of freedom.)

   a. $(Z_1+Z_2)^2 + (Z_3+Z_4)^2$.

   b. $X / \sqrt{\sum (Z_i - X)^2}$ where $X = (Z_1 + Z_2 + \ldots + Z_5)/5$ and the summation goes from $i = 1$ to 5.

   c. Express the following probability in terms of a standard tabulated variable; i.e., your answer might be something like: $P(F(3,4) > 6)$.

   $P((Z_1^2 + Z_2^2 + \ldots + Z_6^2)/(Z_10^2 + Z_11^2 + Z_12^2 + Z_13^2) > 9.24)$.

2. Let $X_1, X_2, X_3$ have joint density

$$f(x_1,x_2,x_3) = \begin{cases} 6 & \text{if } 0 < x_1 < x_2 < x_3 < 1 \\ = 0 & \text{otherwise} \end{cases}$$

Let $Y_1 = X_1/X_2$, $Y_2 = X_2/X_3$ and $Y_3 = X_3$.

   a. Find the joint density of $(Y_1, Y_2, Y_3)$.

   b. Find the expected value of $(X_1/X_3)^3$.

3a. Let $X_1, X_2, \ldots, X_n$ be an independent random sample from a gamma density with $\alpha = 5$ and unknown $\lambda$ (so that the mean of the $X_i$’s is $\alpha / \lambda = 5 / \lambda$). Find the maximum likelihood estimator of $\lambda$.

   b. Find the form of the likelihood ratio test for testing $H_0: \lambda = 1$ against $H_1: \lambda = 2$. (Express the rejection criterion in terms of the value of an easily computable statistic whose distribution is known.)

   c. Is the test of part b uniformly most powerful for testing $H_0$ against the composite hypothesis $H_1: \lambda > 1$? Explain
4. Let $X, Y$ be an independent random sample (of size 2) from a continuous uniform distribution on the real interval $[0, \theta]$ where $\theta (>0)$ is unknown. Consider the following test of $H_0: \theta = 1$ against $H_1: \theta > 1$: reject $H_0$ if either $X$ or $Y$ is greater than 1.

a. Find the probability of a type I error.

b. Find the probability of a type 2 error if $\theta = 2$.

c. Find a formula for the power function of this test and sketch a graph.

5. For positive integers $n$, let $X_n$ have an exponential distribution with mean $1/n$.

a. Write down explicit formulas for the density function $f_n$ and the cumulative distribution function $F_n$ of $X_n$.

b. For each real number $x$, find the limit (if it exists) of $F_n(x)$, as $n \to \infty$.

c. Do the $X_n$’s converge in distribution? If so to what? If not why not? Explain.

6. A random sample of size 5 from a Poisson distribution was 4, 1, 2, 2, 3. The mean $\lambda$ of this Poisson distribution had a prior density that was gamma with $\alpha = 2$ and $\beta = 2$ (so that the mean, $\alpha \beta$, of the prior density is 4). Find the posterior density of $\lambda$ and the mean of the posterior density (i.e., the Bayes estimator of $\lambda$, based on a squared error loss function).

7. Let $X$ have density $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, where $\lambda > 0$.

a. Find the Fisher information for $\lambda$ in an independent random sample of size $n$.

b. What is the Cramer-Rao lower bound for the minimum variance of any unbiased estimator of $\lambda$ based on a sample of size $n$?

c. Find a sensible unbiased estimator of $\lambda$ based on a sample of size $n$. 