Answer all questions!

1. Suppose $X$ has gamma distribution with parameters $\alpha_1$ and $\beta$; $Y$ has gamma distribution with parameters $\alpha_2$ and $\beta$, and $X$ and $Y$ are independent.
   a. Use the probability density function of $X$ to find the moment generating function of $X$.
   b. Find the expected value and variance of $X$.
   c. What kind of distribution does $X + Y$ have? Justify!

2. Suppose $X_1, \ldots, X_n$ is a random sample from a distribution with probability density function
   
   $$f(x; \theta) = \begin{cases} \frac{\Gamma(3\theta)}{\Gamma(\theta)\Gamma(2\theta)} x^{\theta-1} (1-x)^{2\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find, with justification, a sufficient statistic for $\theta$.

3. Suppose a random sample $X_1, \ldots, X_{100}$ is drawn from a distribution with probability density function
   
   $$f(x; \theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } \theta = 0 \\ 6x(1-x) & \text{if } 0 < x < 1 \text{ and } \theta = 1 \\ 0 & \text{if } x \leq 0 \text{ or } x \geq 1. \end{cases}$$

   Describe as best you can the best critical region of size $\alpha$ for testing the hypothesis $\theta = 0$ against the alternative hypothesis $\theta = 1$.

4. Suppose $X_1$ and $X_2$ are independent random variables, $X_1$ has $\Gamma(\alpha, 1)$ distribution, and $X_2$ has $\Gamma(\beta, 1)$ distribution. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$. Find the joint probability density function of $Y_1$ and $Y_2$. Are $Y_1$ and $Y_2$ independent?

5. The standard decision theory approach to the estimation of an unknown parameter $\theta$ introduces the loss function $L(\theta, a)$ which is the cost of deciding that the parameter has the value $a$ when it is in fact equal to $\theta$. The estimate $a$ can be chosen to minimize the posterior expected loss,

   $$E[L(a|y)] = \int L(\theta, a)p(\theta|y) \, d\theta.$$ 

   This optimal choice of $a$ is called a Bayes estimate for the loss function $L$. Let $k_0$ and $k_1$ be nonnegative numbers, not both zero, and define $L(\theta, a)$ to be $k_0(\theta - a)$ if $\theta \geq a$ and otherwise it is $k_1(a - \theta)$. Show that any $k_0/(k_0 + k_1)$ quantile of the posterior distribution is a Bayes estimate of $\theta$ for this loss.

6. Suppose that data $(x_1, \ldots, x_k)$ follow a multinomial distribution with parameters $(\theta_1, \ldots, \theta_k)$. Suppose that $\theta = (\theta_1, \ldots, \theta_k)$ has a Dirichlet prior distribution (i.e. the natural conjugate prior for the multinomial). Let $\alpha = \theta_1/(\theta_1 + \theta_2)$. Find the marginal posterior distribution for $\alpha$ and show that this distribution is identical to the posterior distribution for $\alpha$ obtained by treating $x_1$ as an observation from the binomial distribution with probability $\alpha$ and sample size $x_1 + x_2$, ignoring the data $x_3, \ldots, x_k$. 

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