Do as many problems as possible!

1. With prior probability $1/4$, $\Theta = 2$; otherwise, $\Theta = 1.3$. Draw a random sample of size 3 from a Poisson distribution with parameter $\Theta$. If the 3 observations are 2, 3, and 1, then what is the posterior probability that $\Theta = 2$?

2. A random sample of size 4 is to be drawn from a normal distribution with unknown mean $\mu$ and unknown variance $\sigma^2 > 0$. Describe as best you can a test at significance level 0.05 of the hypothesis $\mu = 100$ against the alternative hypothesis $\mu \neq 100$. Call the observed values $x_1, x_2, x_3,$ and $x_4$. To actually perform the test, you would need a value from a commonly available table; carefully describe how you would find this value if you had the table available.

3. Suppose $X_1$ and $X_2$ are independent random variables so that $X_1$ has $\Gamma(\alpha_1, 1)$ distribution and $X_2$ has $\Gamma(\alpha_2, 1)$ distribution. Suppose $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$. Find the joint probability density function of $Y_1$ and $Y_2$ and the marginal probability density function of $Y_2$.

   Hint: Recall that if $X$ has $\Gamma(\alpha, \beta)$ distribution, then its probability density function is

   $$
   f(x) = \begin{cases} 
   \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta} & \text{if } x > 0 \\
   0 & \text{otherwise.}
   \end{cases}
   $$

4. Find the moment generating function for a random variable $X$ with $\Gamma(\alpha, \beta)$ distribution. Use this function to find $E(X)$ and $\text{Var}(X)$. 
5. Suppose $X_1, \ldots, X_n$ form a random sample from a distribution given by the probability density function

$$f(x; \theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Describe as best you can the best critical region of size $\alpha$ for testing the hypothesis $H_0 : \theta = 3$ against the alternative hypothesis $H_1 : \theta = 1$.

6. Suppose $X_1, \ldots, X_n$ form a random sample from the uniform distribution on the interval $(0, \theta]$ where $\theta > 0$. Find the maximum likelihood estimator of $\theta$. Justify your answer.

7. A random sample of size 25 is drawn from a normal distribution with unknown mean $\mu$ and variance $\sigma^2 = 10$. The observed mean is 8.54. Find a 95% confidence interval for $\mu$. 

*Hint:* You will need a value $b$ such that $P(\left|Z\right| < b) = 0.95$ where $Z$ is a standard normal random variable. You should know approximately the value of $b$. 

2