1. Find an explicit conformal map from the region

\[ \{ z : |z| < 1 \} - \{ x \in \mathbb{R} : x \leq 0 \} \]

onto the upper halfplane \( \{ \text{Im } z > 0 \} \).

2. Find the explicit Laurent series of the function

\[ f(z) = \frac{1}{z(z-3)} \]

on the annulus \( \{ z : 1 < |z-1| < 2 \} \) centered at 1.

3. Let \( D \subset \mathbb{C} \) be open and connected, and fix \( z_0 \in D \); set \( A(D, z_0) = \{|f'(z_0)| : f \text{ holomorphic on } D \text{ and } |f(z)| < 1 \text{ for } z \in D \} \). Prove that \( A(D, z_0) \) is a compact subset of \( \mathbb{R} \). What is \( A(\mathbb{C}, z_0) \)?

4. Let \( f \) be holomorphic in the connected region \( \Omega \subset \mathbb{C} \), and assume that there exists a nonempty open set \( U \subset \Omega \), such that \( |f(z)| = 1 \) for all \( z \in U \). Show that \( f \) is constant in \( \Omega \).

5. Suppose \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is holomorphic on the closed unit disc. Prove that

\[ \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 . \]
6. Suppose $h$ is holomorphic in a neighborhood of $\{z : |z| \leq R\}$, and that $h(z) \neq 0$ for $|z| = R$.

(a) Use the Theorem of Residues to show that

$$\oint_{|z| = R} \frac{h'(z)}{h(z)} \, dz = 2\pi i \, Z_R(h) ,$$

where $Z_R(h)$ is the number of zeroes of $h$ in $\{|z| < R\}$, counted with multiplicities.

(b) Use (a) to prove that if $f$ and $g$ satisfy the same hypotheses as $h$, and if

$$|f - g| < |f| \text{ on } \{|z| = R\} ,$$

then $Z_R(f) = Z_R(g)$.

7. Use the Theorem of Residues for appropriate contours to evaluate

$$\int_{-\infty}^{\infty} \frac{\sqrt{x+i}}{1+x^2} \, dx ,$$

where on $\{\text{Im } z > 0\}$, we choose the branch of $\sqrt{z+i}$ whose value at 0 is $e^{\pi i/4}$. Describe your method carefully, and include verification of all relevant limit statements.

8. Find an explicit series representation for a meromorphic function on $\mathbb{C}$, which is holomorphic on $\mathbb{C} - \{1, 2, 3, \ldots\}$, and which has at each point $z = n \in \mathbb{N}$ a simple pole with residue $n$. Include proofs of all required convergence statements.

9. Prove that all holomorphic automorphisms of $\mathbb{C}$ (i.e. holomorphic maps $f : \mathbb{C} \to \mathbb{C}$ which are one-to-one and onto) are precisely the linear functions $f(z) = a + bz$ for arbitrary $a, b \in \mathbb{C}$. 