Ph.D. Prelim in Complex Analysis

January 18, 1994

1. Let \( f \) be analytic in the unit disk \( D \). Use Cauchy’s integral formula to establish the power series representation of \( f \) in \( D \). Obtain both an integral formula and a derivative formula for the \( n \)-th coefficient.

2. Let \( \Omega \) be a region and let \( \mathcal{F} = \{ f : f \text{ is analytic in } \Omega \text{ and } |f(z)| \leq 1, \forall z \in \Omega \} \). Fix \( z_0 \in \Omega \) and show that \( \exists g \in \mathcal{F} \) such that \( \text{Re } g'(z_0) \geq \text{Re } f'(z_0), \forall f \in \mathcal{F} \).

3. Let \( f \) be analytic and nonconstant in a region \( \Omega \) with \( \mu = \text{Re } f \) and \( v = \text{Im } f \).
   
   (a) Show that \( |f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2 \).

   (b) Determine all real numbers \( a \) and \( b \) such that \( au^2 + bv^2 \) is harmonic in \( \Omega \).

4. Let \( \Omega = \{ z : |z - i| < 1 \} \) and \( H = \{ z : \text{Im } z > 0 \} \). Map \( H \setminus \overline{\Omega} \) conformally onto \( \Omega \).

5. If \( p \) is a polynomial, prove that the series \( \sum_{n=0}^{\infty} p(n)z^n \) defines a rational function.
   
   **HINT:** Note that any linear combination of rational functions is a rational function.

6. (a) Let \( f \) be analytic in the unit disk \( D \) with \( \lim_{|z| \to 1^-} f(z) = 0 \).

   Prove \( f \equiv 0 \).

   (b) Let \( g \) be analytic in \( D \). Prove that the statement \( \lim_{|z| \to 1^-} g(z) = \infty \) is impossible.

7. Let \( f \) be meromorphic in \( \mathbb{C} \) and bounded outside of some circle. Determine the form of \( f \) as completely as possible.

8. Let \( \Gamma = \{ z : |z| = 1 \} \).
   
   (a) Show that the mapping \( z \mapsto (z + 1)^2 \) takes \( \Gamma \) onto the cardioid \( r = 2(1 + \cos \theta) \). Sketch this cardioid.

   (b) Let \( g(w) = \int_{\Gamma} \frac{z(z+1)}{z^2 + 2z - w} \, dz \) (\( \Gamma \) traversed once counterclockwise). Use the result of part (a) to sketch a domain containing \( 0 \) on which \( g \) is analytic.

   (c) Determine \( g(0) \) and \( g'(0) \).