\textbf{COMPLEX ANALYSIS Preliminary Exam}

Jan. 2015

\( \mathbb{D} \) denotes the open unit disc \( \{ z \in \mathbb{C} : |z| < 1 \} \), \( \overline{\mathbb{D}} \) denotes the closed unit disc \( \{ z \in \mathbb{C} : |z| \leq 1 \} \), and \( \partial \mathbb{D} \) denotes the unit circle \( \{ z \in \mathbb{C} : |z| = 1 \} \).

Make sure to show all your work!

1a) Let \( f : \Omega \rightarrow \mathbb{C} \) where \( \Omega \subseteq \mathbb{C} \) is open. State the definition of \( f \) being complex differentiable at a point \( z_0 \in \Omega \), and state the definition of \( f \) being holomorphic at \( z_0 \in \Omega \).

b) Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be given by \( f(z) = (\pi)^2 \). For what values of \( z_0 \) is \( f \) complex differentiable at \( z_0 \)? For what values of \( z_0 \) is \( f \) holomorphic at \( z_0 \)?

2) Find the number of roots (counting multiplicity) that \( f(z) = z^7 + z^5 - 8z^3 + 2z + 1 \) has between the circles \( \{ z \in \mathbb{C} : |z| = 1 \} \) and \( \{ z \in \mathbb{C} : |z| = 2 \} \).

3) Use the Residue theorem to show that \( \int_{-\infty}^{\infty} \frac{e^{\frac{1}{4}x}}{1 + e^x} \, dx = \pi \sqrt{2} \). \textit{Hint:} use a rectangular contour with bottom edge on the line \( \{ z \in \mathbb{C} : \text{Im}z = 0 \} \) and upper edge on the line \( \{ z \in \mathbb{C} : \text{Im}z = 2\pi \} \).

4) Let \( f \) be holomorphic on open set containing \( \overline{\mathbb{D}} \). Use the maximum modulus principle to prove that there exists \( z_0 \in \partial \mathbb{D} \) such that

\[ \left| \frac{1}{z_0} - f(z_0) \right| \geq 1. \]

5) Find the Laurent series or \( f(z) = \frac{1}{(z-1)(z-2)} \) inside the region \( 1 < |z-3| < 2. \)
6) Prove Hurwitz’s theorem: Let $\Omega$ be open and connected and let $\{f_k\}$ be a sequence of holomorphic functions on $\Omega$ that converges normally to $f$ on $\Omega$. If $f$ is non constant on $\Omega$ and has a zero of order $n$ at $z_0$, then for large $k$, $f_k$ has precisely $n$ zeros (counting multiplicity) in a small neighborhood of $z_0$. *hint:* Argument principle!

7) Let $f$ be a holomorphic function on an open connected subset $\Omega$ of $\mathbb{C}$ and assume that $f$ is not identically zero on $\Omega$. If $\mathcal{Z} = \{z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \ldots\}$ then prove (without using the identity principle) that $\mathcal{Z} = \emptyset$.

8) Let $f : \mathbb{D} \to \mathbb{D}$ with $f(z_0) = c_0$ for $z_0, c_0 \in \mathbb{D}$. Using the Schwarz lemma, prove that

$$\left| \frac{f(z) - c_0}{1 - c_0 f(z)} \right| \leq \left| \frac{z - z_0}{1 - z_0 \bar{z}} \right|$$

for all $z \in \mathbb{D}$. 