1. Suppose the complex valued function $f$ is continuously differentiable in the real sense with respect to $(x, y)$ in a neighborhood $U$ of $0 \in \mathbb{C}$, where $z = x + iy$.

   a) State the Cauchy-Riemann equations for $f$ at a point $(x, y) \in U$.

   b) Show in detail that $f$ has a complex derivative at $0$ (i.e., $\lim_{z \to 0} \frac{f(z) - f(0)}{z}$ exists) if and only if $f$ satisfies the Cauchy-Riemann equations at $0$.

   c) Show that $f(z) = z^2 \overline{z}$ has a complex derivative at $0$ but is not holomorphic in any neighborhood of $0$.

2. Suppose $f$ is a bounded holomorphic function on the open unit disc $D$. Show that

\[
(1 - |z|) |f'(z)| \leq \sup_{z \in D} |f(z)|
\]

for all $z \in D$.

3. a) Show that the function $u(x, y) = \log(x^2 + y^2)$ is harmonic on $\mathbb{C} - \{0\}$.

   b) Find a holomorphic function $h$ on $G = \{z : \text{Re } z > 0\}$ so that $u = \text{Re } h$ on $G$.

   c) Determine the imaginary part of $h$ on $G$.

   d) Is $u$ the real part of a holomorphic function on $\mathbb{C} - \{0\}$? Justify your answer.

4. a) State Rouché’s Theorem.

   b) Determine the number of zeroes of the polynomial

   \[
p(z) = z^7 - z^5 + 6z^3 - 2z + 1
\]

   inside the annulus $\{z : 1 < |z| < 2\}$.

5. a) State the Residue Theorem.

   b) Suppose $0 < a < 1$. Prove that

   \[
   \int_{0}^{\infty} \frac{x^{-a}}{1 + x} \, dx = \frac{\pi}{\sin(\pi a)}
   \]

6. Find the Laurent series of the function

   \[
f(z) = \frac{1}{(z + 1)(z - 2)}
\]

   on the annulus $\{z : 1 < |z - 1| < 2\}$. 

1
7. Let $f$ be holomorphic on $\mathbb{C}$ and suppose $P$ is a polynomial in $z$, so that for some constant $M$ one has

$$|f(z)| \leq M |P(z)| \quad \text{for all} \quad z \in \mathbb{C}. $$

Show that there exists a constant $C$ so that $f(z) = CP(z)$ for all $z \in \mathbb{C}$.

8. Let $\{f_n : n = 1, 2, 3, \ldots\}$ be a uniformly bounded sequence of holomorphic functions on $D$. Suppose there exists a point $a \in D$, so that $\lim_{n \to \infty} f_n^{(k)}(a) = 0$ for each $k = 0, 1, 2, \ldots$ $(f_n^{(k)}$ is the $k$th derivative of $f_n$). Show that $f_n \to 0$ uniformly on each compact subset of $D$. 