PHD PRELIM EXAM: COMPLEX ANALYSIS  
August 2012

\( \mathbb{D} \) is the open unit disc centered at 0. Holomorphic means the same as (complex) analytic.

1. a) Let \( D \subset \mathbb{C} \) be a disc and suppose that \( f : D \to \mathbb{C} \) is a continuous function such that \( \int_{\partial \Delta} f(z)dz = 0 \) for every closed triangle \( \Delta \subset D \). Show that there exists a holomorphic function \( H \) on \( D \) such that \( H'(z) = f(z) \) on \( D \). (Do not just quote a theorem. You must give some expression that defines \( H \) and verify in detail that \( H' = f \).

b) State Morera’s Theorem and explain how the result in a) leads to a proof of it.

2. Suppose \( f \) is a continuous function on the domain \( \Omega \subset \mathbb{C} \) such that for some integer \( n \geq 2 \) the function \( f^n \) is holomorphic on \( \Omega \). Does it follow that \( f \) is holomorphic on \( \Omega \)? Give proof or discuss a counterexample!

3. Let \( D \subset \mathbb{C} \) be a simply connected domain, and let \( f \) be a non-vanishing complex-valued harmonic function in \( D \). Prove that if there is a branch of \( \log f(z) \) which is also harmonic, then either \( f(z) \) or its complex conjugate \( \overline{f(z)} \) is holomorphic on \( D \).

4. Find the Laurent series of

\[
f(z) = \frac{1}{z(1-z^2)}
\]

on the region \( G = \{ z \in \mathbb{C} : |z-1| > 2 \} \).

5. Let \( f \) be a holomorphic function on the open unit disc \( \mathbb{D} \). Suppose that there is an \( 0 < r < 1 \) such that the restriction of \( f \) to the annulus

\[
U = \{ z \in \mathbb{C} : r < |z| < 1 \}
\]

is one-to-one. Prove that \( f \) is one-to-one on \( \mathbb{D} \).

6. a) State the Schwarz Lemma for functions holomorphic on the unit disc \( \mathbb{D} \) and prove it. (You may assume power series expansion and max. mod. principle.)

b) Use a) to prove: If \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic, then

\[
\frac{|f(z) - f(w)|}{1 - f(z)f(w)} \leq |z - w| \frac{|1 - z\overline{w}|}{1 - |z||\overline{w}|}
\]

for all \( z, w \in \mathbb{D} \).

7. Show that

\[
\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} \, dx = \pi (\sqrt{2} - 1).
\]

8. Give a precise formula (via an integral, or infinite sum, or product, etc.) for a meromorphic function on \( \mathbb{C} \) which has simple poles at all points \( n = 1, 2, 3, ..., \) with residue \( n \) at \( n \) and which is holomorphic at all other points, and prove in detail that your formula satisfies all requirements. (You may use an appropriate convergence theorem for holomorphic functions.)