1. a) State the Casorati-Weierstrass theorem concerning essential singularities;
   b) State and prove Laurent’s series decomposition theorem.

2. Prove the following uniqueness theorem for real-valued harmonic functions:
   Suppose that \( h(x, y) \) is a real-valued harmonic function in a domain \( D \) and suppose that \( U \subset D \) is an open subset in \( D \). If \( h \) vanishes on \( U \), it vanishes at every point in \( D \).

3. Let \( f(z) \) be entire, and \( g(z) \) be an analytic function in a neighborhood of \( z = 1 \) which satisfies
   \[
   g^{(n)}(1) = \frac{(f^{(n)}(1))^{\alpha}}{(n!)^{\alpha-1}}, \quad \alpha > 0.
   \]
   Prove that \( g \) can be extended to an entire function.

4. Find a conformal mapping of the upper half plane with a slit from 0 to \( i \) along the imaginary axis onto the unit disk.

5. Suppose \( f(z) \) is analytic in \( |z| \leq 1 \), \( f(0) = 0 \), and \( |f(z)| \leq |a - z| \) for \( |z| = 1 \), where \( |a| > 1 \). Show that \( |f'(0)| \leq |a| \).

6. Prove that
   \[
   \int_{-\infty}^{\infty} \frac{x \sin ax}{x^2 + 1} \, dx = \begin{cases} 
   \pi e^{-a} & a > 0 \\
   -\pi e^a & a < 0.
   \end{cases}
   \]

7. Prove that the equation \( z^3 e^z = 1 \) has infinitely many complex solutions. How many of them are real?
8. Let $D$ be a bounded domain, and let $f(z)$ be an analytic function from $D$ into $D$. Show that if $z_0$ is a fixed point for $f(z)$, then $|f'(z_0)| \leq 1$. 

*Hint:* Use the argument principle.