Let $\mathbb{C}$ denote the complex plane and $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ denote the unit disk. A holomorphic function is just another name for a (complex) analytic function.

1. Find ALL values of $i^i$ and $\sqrt{i}$ in the form $x + iy$ or $(x, y)$, $x, y \in \mathbb{R}$.

2. Find the Laurent series of the function 

$$f(z) = \frac{z}{(z-1)(z-2)}$$

in the following regions: (1) $0 < |z-1| < 1$, (2) $|z-2| > 1$.

3. Let $f(z)$ be a holomorphic function in $\mathbb{D}$ which extends continuously to $\overline{\mathbb{D}}$ and $dA$ be area measure. Show that

$$f(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(w)\,dA(w)}{(1-z\,\bar{w})^2}$$

for all $z \in \mathbb{D}$.

4. Suppose $f(z)$ is an entire function and the real part of $f(z)$ is never zero. Show that $f$ must be a constant.

5. a) State the Schwarz Lemma for $\mathbb{D}$ and prove it, assuming power series expansion and maximum modulus principle.

b) Show that every $h \in Aut(\mathbb{D})$ (i.e. biholomorphic map of $\mathbb{D}$ onto itself) is of the form

$$h(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for some $\theta \in [0, 2\pi]$ and $a \in \mathbb{D}$. You may use the Schwarz Lemma, but must prove all other assertions you make.

6. Let $G \subset \mathbb{C}$ be open and simply connected, and $A \subset G$ a discrete subset of $G$. Prove that a holomorphic function $f$ on $G \setminus A$ has an antiderivative on $G \setminus A$ (i.e., there is $F$ holomorphic on $G \setminus A$ with $F' = f$ on $G \setminus A$) if and only if $res_a(f) = 0$ for all $a \in A$. 
7. Let \( f(z) \) be a holomorphic function in \( D \) which extends continuously to \( \overline{D} \) which satisfies \( |f(z)| < 1 \) for all \( z \in \partial D \). Show that there is exactly one point \( w \in D \) such that \( f(w) = w \).

8. Let \( \mathcal{F} = \{ f : f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ with } |a_n| \leq n \text{ for all } n = 0, 1, 2, \ldots \} \).
   a) Prove that every \( f \in \mathcal{F} \) defines a holomorphic function on \( D \).
   b) Prove that \( \mathcal{F} \) is a compact subset of the set of all holomorphic functions on \( D \) in the topology of uniform convergence on compact subsets of \( D \).

9. Let \( \Gamma = \{ \omega \in \mathbb{C} : \omega = m + in \text{ for all } m, n \in \mathbb{Z} \} \). Carefully prove that the series
\[
P(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma, \omega \neq 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)
\]
defines a meromorphic function on \( \mathbb{C} \). Identify the region where \( P(z) \) is holomorphic.