Complex Analysis Prelim. (Jan. 2009)

In the following, \( \mathbb{D} \) stands for the open unit disk, \( \mathbb{C} \) stands for the complex plane.

**Part 1.** Do all of the following problems.

1. Show that a complex polynomial of degree \( n > 0 \) has precisely \( n \) zeros in the complex plane.

2. a) Find all solutions of the equation \( z^6 + 1 = 0 \).

   b) Let \( g(z) = z^2 \pi \). Find all points where \( g \) is complex differentiable.

3. Find an explicit conformal map from the region \( G = \mathbb{D} \setminus \{ 0 \leq x < 1 \} \) onto the unit disc \( \mathbb{D} \).

4. (a) State and prove the Liouville’s Theorem.

   (b) Let \( V \) be the set of entire functions \( f \) such that \( |f(z)| \leq C|z|^5 \) for some constant \( C \) (depending probably on \( f \)), determine what type of functions are in \( V \) and find the dimension of \( V \).

5. Use residue theory to compute

\[
\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}.
\]
Part 2. Do at least two of the following problems.

6. Let \( f(z) = u(z) + iv(z) \) be holomorphic in a neighborhood of the closed unit disc \( \mathbb{D} \), where \( u \) and \( v \) are the real and, respectively, the imaginary part of \( f \). Prove the Schwarz formula:

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta + iv(0), \quad |z| < 1.
\]

7. Find all entire functions \( f \) such that \( |f(z)| = 1 \) when \( |z| = 1 \).

8. Give as simple as possible a (product) formula for an entire function \( F \) which has a zero of order 1 at each point \( c_n = \sqrt{n}, \ n = 1, 2, 3, \ldots \) and no other zero in \( \mathbb{C} \).

9. Find an “explicit” series expansion for a meromorphic function \( f \) on \( \mathbb{C} \) which has a simple pole with residue \( n \) at each positive integer \( n = 1, 2, 3, \ldots \), and is holomorphic at all other points. Be sure to prove all relevant convergence statements.

10. Let \( V = \{ f \in \mathcal{O}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } |a_n| \leq n^2 \text{ for all } n \} \). Prove that there exists \( h \in V \), such that \( |f'(1/2)| \leq |h'(1/2)| \) for all \( f \in V \).