1. a) Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in D$. Prove that for $0 < r < 1$, the series converges absolutely and uniformly on $\{ |z| \leq r \}$.

b) Show that for any positive integer $k$ the power series $\sum_{n=1}^{\infty} n^k z^n$ has radius of convergence $1$ and that its limit equals a rational function on $D$.

2. How many zeroes does $P(z) = 1 + 3 z^8 - z^{16}$ have in the unit disc $D$? Determine the multiplicities of these zeroes!

3. Evaluate
   a) $\int_{\gamma} (z^2 + 3\overline{z}) \, dz$, where $\gamma$ is the upper half of the unit circle from $-1$ to $+1$.
   b) $\oint_{|z|=4} \frac{1}{\sin z} \, dz$, where the circle is traversed once counterclockwise.
   c) $\int_{-\infty}^{\infty} \frac{\cos(\alpha x) \, dx}{1 + x^2}$, where $\alpha$ is real.

4. Suppose $h$ is a holomorphic function on $D$ which satisfies $|h(z)| \leq \frac{1}{1-|z|}$ for all $z \in D$. Show that $|h'(0)| \leq 4$.

5. Let $g$ be the holomorphic function defined in a neighborhood of $i$ as the branch of $\sqrt{1 - z^2}$ which satisfies $g(i) = \sqrt{2}$.
   a) Show that $g$ can be continued analytically along any curve in $G = \mathbb{C} \setminus \{-1, 1\}$.
   b) Can $g$ be continued analytically to define a holomorphic function on $G$? Why?
   c) Show that the analytic continuation of $g$ leads to a holomorphic function on $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

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DENOTES THE OPEN UNIT DISC $\{ z \in \mathbb{C} : |z| < 1 \}$.

Holomorphic functions are also called analytic functions.

Make sure to show all your work!
6. Let \( \{ f_n(z), n = 1, 2, \ldots \} \) be a uniformly bounded sequence of holomorphic functions on \( D \) (i.e., there is \( C < \infty \) such that \( |f_n(z)| \leq C \) for all \( z \in D \) and \( n \)). Suppose there is a point \( a \in D \) such that for each \( k = 0, 1, 2, \ldots \) one has \( \lim_{n \to \infty} f_n^{(k)}(a) = 0 \). (\( f_n^{(k)} \) is the \( k \)th derivative of \( f_n \).) Show that \( f_n \to 0 \) uniformly on each compact subset of \( D \).

7. Characterize all holomorphic functions \( f(z) \) in \( D \) such that \( |f(z)| \leq |\cos(1/z)| \) for all \( z \in D \).

8. a) Prove that the infinite product

\[
P = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right)
\]

converges.

b) Prove that value of \( P \) defined in a) equals

\[
\frac{e^\pi - e^{-\pi}}{2\pi}.
\]

(Hint: You may use an appropriate formula for the sine function.)

9. Find a conformal map \( f \) from the strip \( S = \{ z : |\text{Re} z| < \pi \} \) onto the unit disc \( D \) which satisfies \( f(0) = 0 \). (You may leave the answer as a composition of explicit functions.)

10. Let the complex numbers \( \omega_1 \) and \( \omega_2 \) be linearly independent over \( \mathbb{R} \), and let \( L = \{ \omega = m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \} \).

a) Carefully prove that the series

\[
F(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in L} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}
\]

defines a meromorphic function on \( \mathbb{C} \), and describe the poles and their principal parts.

b) Prove that the function \( F \) defined in a) has periods \( L \), i.e., for any \( \omega \in L \) one has

\[
F(z + \omega) = F(z) \quad \text{for all } z \notin L.
\]

(Hint: Take derivatives!)