1. Let $K$ be a field, $L$ a field extension of $K$. An element $\alpha$ in $L$ is algebraic over $K$ if $\alpha$ is the root of some monic polynomial with coefficients in $K$.

Show that if $\alpha$ and $\beta$ in $L$ are algebraic over $K$, then $\alpha\beta$ is algebraic over $K$.

2. Let $w$ be a primitive cube root of unity. Let $R = \mathbb{Z}[w]$. Let $\lambda = 1 - w$. Show that $R/\lambda R \cong \mathbb{Z}/3\mathbb{Z}$.

3. Let $G$ be the group of $2 \times 2$ invertible matrices of determinant 1 with coefficients in the field of 3 elements.

(a) Show that $G$ has order 24.

(b) Find the number of 3-Sylow subgroups of $G$.

4. Let $G$ be a finite $p$-group, $p$ prime, $V$ a finite dimensional vector space over the field $\mathbb{F}_p$ of $p$ elements. Suppose $G$ acts linearly on $V$ (i.e. there is a homomorphism from $G$ into the group $GL(V)$ of invertible linear transformations from $V$ to $V$). Prove that $G$ has a non-zero fixed point: that is, there is some $\alpha \neq 0$ in $V$ so that $\sigma(\alpha) = \alpha$ for all $\sigma$ in $G$.

5. Let $L/K$ be a Galois extension of fields with Galois group $G$. Let $L = K[\alpha]$. Define $tr(\alpha) = \sum_{\sigma \in G} \sigma(\alpha)$. Let $T_\alpha : L \to L$ be the $K$-linear transformation defined by $T_\alpha(\beta) = \alpha\beta$. Show that $tr(\alpha)$ is the trace of the linear transformation $T_\alpha$.

6. Prove that for any prime $p$, there are at least four isomorphism classes of groups of order $p^3$. 
7. A \( \mathbb{Z} \)-module \( M \) is flat if for any short exact sequence \( 0 \to A \to B \to C \to 0 \) of \( \mathbb{Z} \)-modules, the sequence \( 0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0 \) is exact.

(a) State and prove a criterion for flatness as follows: \( M \) is flat if and only if for any homomorphism \( f : E \to F \) of \( \mathbb{Z} \)-modules, if \( t \) is injective, then \( M \otimes f \) is injective.

(b) Give an example of a non-flat \( \mathbb{Z} \)-module.

8. Let \( K \) be a field, \( M \) a \( K \)-vector space. Let \( M^* = \text{Hom}_R(M, K) \). Show that the canonical map \( M \to M^{**} \) is surjective if and only if \( M \) is finite dimensional.