1. Show that every group of order 42 has each of the following:
   (a) a normal subgroup of order 21;
   (b) a subgroup of order 6.
2. Let $G$ be a finite group containing a subgroup $H$ of index 4. Show that $G$ is not simple.
3. Let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$ be the standard primitive $n$-th root of 1. Let $G$ be the Galois group of $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}$.
   (a) What is the isomorphism class of $G$ as a finite group?
   (b) Describe the elements of $G$ as automorphisms of $\mathbb{Q}(\zeta_5)$. How do these elements act on $\zeta_5$?
   (c) What are the subgroups of $G$? Use them to determine all subfields of $\mathbb{Q}(\zeta_5)$. Describe each subfield in the form $\mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{Q}(\zeta_5)$. What is its degree over $\mathbb{Q}$? What is the minimal polynomial of $\alpha$ over $\mathbb{Q}$?
4. Determine the Galois groups of $x^5 - 11$ over $\mathbb{Q}(\zeta_5)$ and over $\mathbb{Q}$.
5. Determine the (integer) invariant factors of the finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{27} \times \mathbb{Z}_5 \times \mathbb{Z}_{25}$. (This is the analogue for abelian groups of rational canonical form.)
6. Give all possible rational canonical forms for matrices in $\text{GL}_2(\mathbb{Z}_3)$.
7. Show that every group of order 90 has a normal 5-Sylow subgroup.