Preliminary Examination in Algebra
August 2009

(1) Determine the number of \( p \)-Sylow subgroups in the symmetric group \( S_p \), where \( p \) is a prime.

(2) Find all similarity classes of matrices in \( M_7(\mathbb{R}) \) with the minimal polynomial \((x-1)(x^2+1)^2\). For each class write its rational canonical form.

(3) Show that there is no simple group of order \( pqr \), where \( p < q < r \) are prime.

(4) Show that \( A \in M_n(k) \), \( k \) is a field, is similar to \( A^T \) (the transpose of \( A \)).

(5) Let \( B \in M_n(\mathbb{Q}) \) such that \( B^5 = 1 \) and no eigenvalue of \( B \) is equal to 1. Show that \( n \) is divisible by 4.

(6) Let \( F \) be a field of characteristic zero. Suppose that \( K/F \) is finite Galois extension with Galois group \( G \). Prove that if \( a \in K \) and \( g(a) - a \in F \) for all \( g \in G \), then \( a \in F \).

(7) Let \( K \) be the splitting field over \( \mathbb{Q} \), in \( \mathbb{C} \), of \( x^4 - 2 \). Determine the Galois group \( Gal(K/\mathbb{Q}) \) and the subfields of \( K \). For each subfield \( F \) of \( K \), give field generators over \( \mathbb{Q} \).