Ph.D. Preliminary Examination in Algebra

August 31, 2006

1. Show that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, if and only if $m$ and $n$ are relatively prime.

2. Let $p$ be a prime number. How many Sylow $p$-subgroups does $S_p$ have?

3. Show that there is no simple group of order 160.

4. Show that $\mathbb{Z}[\sqrt{3}]$ is a UFD.

5. Let $K$ be a finite field.
   (a) Show that there exists a prime number $p$ so that $K$ contains a subfield $F$ isomorphic to the field $\mathbb{F}_p$ of $p$ elements.
   (b) Show that there exists a polynomial $q(x)$ with coefficients in $F$ such that $K$ is isomorphic (as rings) to the ring $F[x]/(q(x))$.
   (c) Show that $K : F$ is Galois.

6. (a) Describe the Galois group $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ and its action on $\mathbb{Q}(\zeta_5)$, where $\zeta_5 = e^{2\pi i / 5}$.
   (b) Determine the minimal polynomial of $\cos(2\pi/5)$ and show that $\cos(2\pi/5) = \frac{-1 + \sqrt{5}}{4}$.
   (c) Find the tower of subfields of $\mathbb{Q}(\zeta_5)$ and express them as fixed subfields of subgroups of $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$.

7. Prove that a left module $M$ over a ring with identity $R$ is simple (i.e., $M \neq 0$ and $M$ has no proper submodules) if and only if $M$ is isomorphic to $R/I$ for some maximal left ideal $I$.

8. If $A$ is an $n \times n$ matrix with entries in a field $k$, show that $A$ is similar to its transpose $A^t$.

9. (a) Define projective module.
    (b) Define injective module.
    (c) Prove or disprove: $\mathbb{Z}$ is an injective $\mathbb{Z}$-module.
    (d) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.