Ph.D. Preliminary Examination in Algebra

June 9, 2000

1. Let $X$ be any set, $F$ any field, and $F^X$ the set of maps from $X$ to $F$. $F^X$ is endowed with the structure of a vector space over $F$ using “pointwise” addition and multiplication by scalars. Prove that a finite sequence $f_1, f_2, \ldots, f_n$ of elements of $F^X$ is linearly independent if and only if there is a finite sequence of elements $x_1, x_2, \ldots, x_n$ in $X$ for which the $n \times n$ determinant $\det (f_i(x_j))$ is non-zero.

2. Find a complete set of representatives for the isomorphism classes of finite abelian groups of order 1001.

3. Let $K$ be the splitting field over the field $\mathbb{Q}$ of rational numbers of the polynomial 
   \[ f(x) = x^5 - x^4 + x^3 - x^2 + x - 1. \]
   (a) What are the possible values for the minimum degree among the irreducible factors of a polynomial of degree 5?
   (b) Write $f$ as the product of factors irreducible over $\mathbb{R}$.
   (c) Write $f$ as the product of factors irreducible over $\mathbb{Q}$.
   (d) What is the degree of $K$ over $\mathbb{Q}$?
   (e) What is the Galois group of $K$ over $\mathbb{Q}$?

4. Show that if two square matrices of the same finite size over a field are similar in a larger field then they must be similar in the original field.

5. Let $F$ be a field, and let $A$ be the quotient ring
   \[ A = F[t, x, y, z]/(tz - xy)F[t, x, y, z] \]
   where $t, x, y, z$ are independent transcendentals over $F$.
   (a) Show that $A$ has no zero divisors.
   (b) Explain briefly why $A$ is Noetherian.
   (c) Is $A$ a unique factorization domain? (Either prove that it is or exhibit an example of something that does not factor uniquely according to the usual criteria for such uniqueness.)

6. Let $E$ be a finite extension of a field $F$.
   (a) Outline an argument for showing that if $F$ is a finite field, then $E$ is a cyclic Galois extension of $F$.
   (b) Provide an example where $F$ is a field of characteristic 5 and $E$ is an extension of $F$ of degree 5 that is not a Galois extension of $F$.
   (c) For any given field $K$ explain how to obtain an extension $F$ of $K$ and a finite extension $E$ of $F$ for which $E$ is a Galois extension of $F$ with Galois group isomorphic to the symmetric group $S_n$ (consisting of the permutations of $n$ objects).

7. Let $\mathbb{F}_3$ denote the field of 3 elements.
   (a) What is the cardinality of 2-dimensional Cartesian space $\mathbb{F}_3 \times \mathbb{F}_3$ over $\mathbb{F}_3$?
   (b) Let $N$ denote cardinality of the group $GL_2(\mathbb{F}_3)$ of linear automorphisms of $\mathbb{F}_3 \times \mathbb{F}_3$.
      Compute $N$.
   (c) Observe that the multiplicative group $\mathbb{F}_3^*$ is the unique group of order 2 and furthermore that:
      i. Multiplication by invertible scalars gives rise to a homomorphism $\phi$ from $\mathbb{F}_3^*$ to $GL_2(\mathbb{F}_3)$.
      ii. The determinant gives rise to a homomorphism $\psi$ from $GL_2(\mathbb{F}_3)$ to $\mathbb{F}_3^*$
      Explain why the kernel of $\psi$ and the cokernel of $\phi$ both have the same cardinality.
   (d) Is the kernel of $\psi$ isomorphic to the cokernel of $\phi$?

8. Prove over any commutative ring (with 1) that two isomorphic free modules of finite rank must have the same rank.