Let $k \in \mathbb{N}$, $k > 3$. Let, further, $(a_l)^k_{l=1} \subset \mathbb{R}$ be an arithmetic progression such that 
$\left(\sqrt{1-a_l^2}\right)^k_{l=1}$ is a geometric progression.
Prove that $a_1 = a_2 = \cdots = a_k$.

Solution:
Fist we observe that since $\sqrt{1-a_l^2}$ is defined for all $l = 1, 2, \ldots, k$, it follows that
\[a_l \in [-1, 1] \text{ for all } l = 1, 2, \ldots, k. \quad (1)\]

Now, since $(a_l)^k_{l=1}$ is an arithmetic progression, it follows that
\[a_l = \frac{a_{l-1} + a_{l+1}}{2} \text{ for all } l = 2, 3, \ldots, k-1, \quad (2)\]

and since $\left(\sqrt{1-a_l^2}\right)^k_{l=1}$ is a geometric progression, we have that
\[1 - a_l^2 = \sqrt{(1 - a_{l-1}^2) (1 - a_{l+1}^2)} \text{ for all } l = 2, 3, \ldots, k-1. \quad (3)\]
Substituting, now, (2) into (3) and simplifying, we obtain that

\[
1 - \left( \frac{a_{l-1} + a_{l+1}}{2} \right)^2 = \sqrt{1 - a_{l-1}^2} \left( 1 - a_{l+1}^2 \right)
\]

and

\[
1 - a_{l-1}^2 + 2a_{l-1}a_{l+1} + a_{l+1}^2 = \sqrt{1 - a_{l-1}^2} \left( 1 - a_{l+1}^2 \right)
\]

4 - a_{l-1}^2 - 2a_{l-1}a_{l+1} - a_{l+1}^2 = 4 \sqrt{1 - a_{l-1}^2 - a_{l+1}^2 + a_{l-1}a_{l+1}^2}

\[
(4 - a_{l-1}^2 - 2a_{l-1}a_{l+1} - a_{l+1}^2)^2 = 16 \left( 1 - a_{l-1}^2 - a_{l+1}^2 + a_{l-1}a_{l+1}^2 \right)
\]

\[
16 + 4a_{l-1}^2a_{l+1}^2 + a_{l+1}^4 - 8a_{l-1}^2 - 16a_{l-1}a_{l+1} - 8a_{l+1}^2 + 4a_{l-1}^3a_{l+1} + 2a_{l-1}a_{l+1}^2 + 4a_{l-1}a_{l+1}^3 = 16 - 16a_{l-1}^2 + 16a_{l+1}^2 + 16a_{l-1}a_{l+1}^2
\]

\[
a_{l-1}^4 + 4a_{l-1}^2a_{l+1}^2 + 4a_{l-1}a_{l+1}^3 + 8a_{l-1}^2 + 8a_{l-1}a_{l+1} = 0
\]

\[
(a_{l-1}^2 - a_{l+1}^2)^2 - 8a_{l-1}a_{l+1}^2 + 4a_{l-1}^2a_{l+1} + 4a_{l-1}a_{l+1}^2 + 8 \left( a_{l-1}^2 + a_{l+1}^2 - 2a_{l-1}a_{l+1} \right) = 0
\]

\[
(a_{l-1}^2 - a_{l+1}^2)^2 + 4a_{l-1}a_{l+1} \left( -2a_{l-1}a_{l+1} + a_{l-1}^2 + a_{l+1}^2 \right) + 8 \left( a_{l-1}^2 - a_{l+1} \right) = 0
\]

\[
(a_{l-1}^2 - a_{l+1}^2)^2 + 4a_{l-1}a_{l+1} \left( a_{l-1} - a_{l+1} \right)^2 + 8 \left( a_{l-1} - a_{l+1} \right)^2 = 0
\]

\[
(a_{l-1}^2 - a_{l+1}^2)^2 \left( (a_{l-1} + a_{l+1})^2 + 4a_{l-1}a_{l+1} + 8 \right) = 0
\]

(4)

Since from (1) we have that \(a_{l-1}a_{l+1} \in [-1, 1]\) for every \(l = 2, 3, \ldots, k - 1\), it follows that \((a_{l-1} + a_{l+1})^2 + 4a_{l-1}a_{l+1} + 8 > 0\) for all \(l = 2, 3, \ldots, k - 1\).

From the last line of (4) we, therefore, obtain that

\[
(a_{l-1} - a_{l+1})^2 = 0 \text{ for } l = 2, 3, \ldots, k - 1,
\]

or, equivalently, that

\[
a_{l-1} = a_{l+1} \text{ for } l = 2, 3, \ldots, k - 1.
\]

Now, from (2) we obtain that

\[
a_l = \frac{a_{l-1} + a_{l+1}}{2} = a_{l-1} = a_{l+1}
\]

for \(l = 2, 3, \ldots, k - 1\), which completes the proof of the assertion.

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