High School Math Problems
2015
Week 7
Problem and Solution

Prove that the altitudes of a triangle intersect in one point.

Proof. Let ABC be an arbitrary triangle. Let D be the point of intersection of the altitudes through the vertices A and C.

Then, from the problem from Week 5, we have that

$$\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CA} \cdot \overrightarrow{BD} = 0.$$

But

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = 0 \text{ since } \overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ are orthogonal}$$

and

$$\overrightarrow{BC} \cdot \overrightarrow{AD} = 0 \text{ since } \overrightarrow{BC} \text{ and } \overrightarrow{AD} \text{ are orthogonal.}$$

Therefore

$$\overrightarrow{CA} \cdot \overrightarrow{BD} = 0$$

and thus the altitude of the triangle ABC through the vertex B passes through D. $\square$