High School Math Problems
2015
Week 33
Problem and Solution

Let a, b, and c be the lengths of the sides of a triangle and let α, β, and γ be the magnitudes of the angles opposite these sides, respectively.

If α, β, and γ form a geometric sequence with common ratio 2, prove that

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}.$$

Solution:

Since $\beta = 2\alpha$, $\gamma = 2\beta = 4\alpha$, and $\alpha + \beta + \gamma = \pi$, it follows that $7\alpha = \pi$ and therefore $\alpha = \frac{\pi}{7}$, $\beta = \frac{2\pi}{7}$, $\gamma = \frac{4\pi}{7}$.

Let, now, R be the circumradius of the triangle. Then, from the Law of Sines, we have that

$$\begin{aligned}
\frac{1}{b} + \frac{1}{c} &= \frac{1}{2R\sin\frac{2\pi}{7}} + \frac{1}{2R\sin\frac{4\pi}{7}} = \frac{1}{2R} \cdot \frac{\sin\frac{4\pi}{7} + \sin\frac{2\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}} \\
&= \frac{1}{2R} \cdot \frac{2\sin\frac{3\pi}{7}\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\sin\frac{4\pi}{7}} = \frac{1}{2R} \cdot \frac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{4\pi}{7}} \\
&= \frac{1}{2R} \cdot \frac{\sin\left(\pi - \frac{3\pi}{7}\right)}{\sin\frac{\pi}{7}\sin\frac{4\pi}{7}} = \frac{1}{2R} \cdot \frac{\sin\frac{4\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{4\pi}{7}} \\
&= \frac{1}{2R} \cdot \frac{1}{\sin\frac{\pi}{7}} = \frac{1}{a}.
\end{aligned}$$