Let \( a, b, \) and \( c \) be the lengths of the sides of a triangle and let \( \alpha, \beta, \) and \( \gamma \) be the magnitudes of the angles opposite these sides, respectively.

If \( \alpha, \beta, \) and \( \gamma \) form a geometric sequence with common ratio 2, prove that

\[
\frac{1}{a} = \frac{1}{b} + \frac{1}{c}.
\]

**Solution:**

Since \( \beta = 2\alpha, \gamma = 2\beta = 4\alpha, \) and \( \alpha + \beta + \gamma = \pi, \) it follows that \( 7\alpha = \pi \) and therefore \( \alpha = \frac{\pi}{7}, \beta = \frac{2\pi}{7}, \gamma = \frac{4\pi}{7}. \)

Let, now, \( R \) be the circumradius of the triangle. Then, from the Law of Sines, we have that

\[
\frac{1}{b} + \frac{1}{c} = \frac{1}{2R} \cdot \frac{1}{2R} + \frac{1}{2R} \cdot \frac{1}{2R} \cdot \sin \left( \frac{\pi - 3\pi}{7} \right) = \frac{1}{2R} \cdot \frac{1}{2R} \cdot \sin \frac{\pi}{7} \cdot \sin \frac{4\pi}{7} = \frac{1}{2R} \cdot \frac{1}{2R} \cdot \frac{1}{2R} \cdot \frac{1}{2R} \cdot \sin \frac{\pi}{7} \cdot \sin \frac{4\pi}{7} = \frac{1}{a}.
\]

©Ivana Alexandrova