# High School Math Problems
# 2015
# Week 25
# Problem and Solution

Solve the equation

$$\sqrt{1+\sqrt{1-x^2}}\left(\sqrt{(1+x)^3}-\sqrt{(1-x)^3}\right)=2+\sqrt{1-x^2}.$$

**Solution:**

The domain of the equation is $D_x=\{x:|x|\le 1\}$. Therefore there exists $\alpha\in[0,\pi]$ such that $x=\cos\alpha$. Substituting into the equation, we then obtain

$$\sqrt{1+\sqrt{1-\cos^2\alpha}}\left(\sqrt{(1+\cos\alpha)^3}-\sqrt{(1-\cos\alpha)^3}\right)=2+\sqrt{1-\cos^2\alpha}$$

$$\sqrt{1+\sin\alpha}\left(\sqrt{(1+\cos\alpha)^3}-\sqrt{(1-\cos\alpha)^3}\right)=2+\sin\alpha$$

$$\sqrt{1+\sin\alpha}\left(\sqrt{\left(2\cos^2\frac{\alpha}{2}\right)^3}-\sqrt{\left(2\sin^2\frac{\alpha}{2}\right)^3}\right)=2+\sin\alpha$$

$$2\sqrt{2}\sqrt{1+\sin\alpha}\left(\cos^3\frac{\alpha}{2}-\sin^3\frac{\alpha}{2}\right)=2+\sin\alpha$$

$$2\sqrt{2}\sqrt{\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos^2\frac{\alpha}{2}+\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\right)=2+\sin\alpha$$

$$2\sqrt{2}\sqrt{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos^2\frac{\alpha}{2}+\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\right)=2+\sin\alpha$$

$$2\sqrt{2}\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(1+\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right)=2+\sin\alpha$$

$$\sqrt{2}\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)(2+\sin\alpha)=2+\sin\alpha$$

$$\sqrt{2}\cos\alpha=1.$$

Therefore $x=\cos\alpha=\dfrac{\sqrt{2}}{2}\in D_x$ is the solution.

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