Chapter 4 Graphs and Matrices

PAD637 Week 3 Presentation
Prepared by
Weijia Ran & Alessandro Del Ponte
Outline

• Graphs: Basic Graph Theory Concepts
• Directed Graphs
• Signed Graphs & Signed Directed Graphs
• Valued Graphs & Valued Directed Graphs
• Generalization: Multigraphs & Hypergraphs
• Mathematical Relations
• Matrices and Basic Matrix Operations for Graphs
Why Graph Theory?

- a vocabulary: to label and denote social structural properties;
- mathematical operations and ideas: to quantify and measure these properties;
- ability: to prove theorems about graphs (representations of social structure).
Graphs
Graphs

**Graph:**
a model for a network with an undirected dichotomous relation;
a set of nodes & a set of lines

**Fig. 4.1.** Graph of “lives near” relation for six children

**Relevant concepts:**
adjacent (two nodes); incident with (a node & a line)
trivial (# of nodes=1); empty (# of nodes ≥ 1 & # of lines=0)
Subgraphs of a graph (p.98)

a.
\[ \mathcal{N} = \{n_1, n_2, n_3, n_4, n_5\} \]
\[ \mathcal{L} = \{l_1, l_2, l_3, l_4\} \]
\[ l_1 = (n_1, n_2) \]
\[ l_2 = (n_1, n_3) \]
\[ l_3 = (n_1, n_5) \]
\[ l_4 = (n_3, n_4) \]

b. subgraph
\[ \mathcal{N}_s = \{n_1, n_3, n_4\} \]
\[ \mathcal{L}_s = \{l_2\} \]

c. subgraph generated by nodes \(n_1, n_3, n_4\)
\[ \mathcal{N}_s = \{n_1, n_3, n_4\} \]
\[ \mathcal{L}_s = \{l_2\} \]

d. subgraph generated by lines \(l_1, l_3\)
\[ \mathcal{N}_s = \{n_1, n_2, n_5\} \]
\[ \mathcal{L}_s = \{l_1, l_3\} \]
Dyads and Triads

Two possible dyadic states: adjacent or not adjacent

Four possible triadic states in a graph
Nodal Degree & Density

Degree of a node: $d(n_i) \quad 0 \leq d(n_i) \leq (g-1)$

Mean nodal degree:

$$\bar{d} = \frac{\sum_{i=1}^{g} d(n_i)}{g} = \frac{2L}{g}$$

Variance of the degrees:

$$S^2_D = \frac{\sum_{i=1}^{g} (d(n_i) - \bar{d})^2}{g}$$

Density of a graph:

$$\Delta = \frac{L}{g(g-1)/2} = \frac{2L}{g(g-1)}$$

$$\Delta = \frac{\bar{d}}{(g-1)}$$
Complete and empty graphs (p.102)

a. Empty 
\( (L = 0) \)
\[ n_1 \quad n_2 \quad n_3 \quad n_4 \quad n_5 \]

b. Complete 
\( (L = g(g-1)/2 = 10) \)
\[ \begin{array}{c}
\quad n_1 \quad n_2 \\
\quad n_3 \quad n_4 \\
\quad n_5 \\
\end{array} \]

c. Intermediate 
\( (0 < L < g(g-1)/2; \text{ here } L = 4) \)
\[ \begin{array}{c}
\quad n_1 \quad n_2 \\
\quad n_3 \\
\quad n_4 \\
\end{array} \]
Walks, trails, and paths

• **Walk**
  – not all nodes/lines
  – some nodes/lines used more than once
  – length of a walk: number of occurrences of lines

• **Trail**
  – a walk in which all lines are distinct

• **Path**
  – a walk in which all nodes and all lines are distinct

• **Reachable**
  – two nodes are reachable if there is a path between them
Walks, trails, and paths in a graph (p.106)

A walk would be $W = n_1 \ l_2 \ n_4 \ l_3 \ n_2 \ l_3 \ n_4$

A trail would be $W = n_4 \ l_3 \ n_2 \ l_4 \ n_3 \ l_5 \ n_4 \ l_2 \ n_1$

A path would be $W = n_1 \ l_2 \ n_4 \ l_3 \ n_2$

\[ n_5 = \text{Joe} \ \\
      \text{---} \ \\
      \text{---} \ \\
      \text{---} \ \\
      \text{---} \ \\
\text{n_1 = Jack} \ \\
\text{---} \ \\
\text{n_4 = Jim} \ \\
\text{---} \ \\
\text{n_3 = Jerry} \]
Closed Walks, Tours, and Cycles

Tour $n_3\ n_2\ n_4\ n_3\ n_5\ n_1\ n_4\ n_3$

Cycles $n_5\ n_1\ n_4\ n_3\ n_5$

$n_2\ n_3\ n_4\ n_2$

$n_2\ n_4\ n_1\ n_5\ n_3\ n_2$

Closed walk $n_5\ n_1\ n_4\ n_3\ n_2\ n_4\ n_1\ n_5$
Connected Graphs and Components (p.109)
Geodesic/Distance, Eccentricity and Diameter
(p.111)

Geodesic distances

\[ d(1,2) = 1 \]
\[ d(1, 3) = 1 \]
\[ d(1,4) = 2 \]
\[ d(1,5) = 3 \]
\[ d(2, 3) = 1 \]
\[ d(2,4) = 1 \]
\[ d(2,5) = 2 \]
\[ d(3, 4) = 1 \]
\[ d(3,5) = 2 \]
\[ d(4,5) = 1 \]

Diameter of graph = \( \max d(i,j) = d(1,5) = 3 \)
Connectivity of Graphs: Cutpoint & k-node cut
(p.112)

Node $n_1$ is a node cut, or cutpoint

The graph without node $n_1$
Connectivity of Graphs: Bridge & I-line cut

Line \((n_2 \ n_3)\) is a bridge

Graph without line \((n_2 \ n_3)\)
Node- and line-Connectivity (p.116)

$n_2$ and $n_4$ comprise a 2-node cut

The graph without $n_2$ and $n_4$
Isomorphic Graphs (p.118)

\[ g^+ \]
- \( \phi(n_1) = Keith \)
- \( \phi(n_2) = Eliot \)
- \( \phi(n_3) = Sarah \)
- \( \phi(n_4) = Allison \)
- \( \phi(n_5) = Drew \)
- \( \phi(n_6) = Ross \)

\[ g \]
- Allison
- Sarah
- Ross
- Keith
- Drew
- Eliot
Special kinds of Graphs (p.119)

- Complement
- Tree and forest
- Bipartite Graphs
Directed Graphs
Friendship at the beginning of the year for six children

<table>
<thead>
<tr>
<th>Actor</th>
<th>Likes at beginning of year</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1$</td>
<td>Allison</td>
</tr>
<tr>
<td>$n_2$</td>
<td>Drew</td>
</tr>
<tr>
<td>$n_3$</td>
<td>Eliot</td>
</tr>
<tr>
<td>$n_4$</td>
<td>Keith</td>
</tr>
<tr>
<td>$n_5$</td>
<td>Ross</td>
</tr>
<tr>
<td>$n_6$</td>
<td>Sarah</td>
</tr>
</tbody>
</table>

$t_1 = (n_1, n_2)$
$t_2 = (n_1, n_5)$
$t_3 = (n_2, n_3)$
$t_4 = (n_2, n_6)$
$t_5 = (n_3, n_2)$
$t_6 = (n_4, n_5)$
$t_7 = (n_5, n_6)$
$t_8 = (n_6, n_2)$
<table>
<thead>
<tr>
<th>Undirected</th>
<th>Directed</th>
</tr>
</thead>
<tbody>
<tr>
<td>nodes</td>
<td>first node (sender)</td>
</tr>
<tr>
<td></td>
<td>second node (receiver)</td>
</tr>
<tr>
<td>line</td>
<td>arc: an ordered pair of nodes</td>
</tr>
<tr>
<td>dyad</td>
<td>null dyad: two nodes without arcs</td>
</tr>
<tr>
<td></td>
<td>asymmetric: with an arc going in one direction</td>
</tr>
<tr>
<td></td>
<td>mutual dyad: with two arcs</td>
</tr>
<tr>
<td>nodal degree</td>
<td>nodal indegree: # of node terminating at</td>
</tr>
<tr>
<td></td>
<td>nodal outdegree: # of arcs originating with</td>
</tr>
<tr>
<td>walk/path/cycle</td>
<td>directed walk/path/cycle: the same direction</td>
</tr>
<tr>
<td></td>
<td>semiwalk/path/cycle: the direction is irrelevant</td>
</tr>
<tr>
<td>Undirected</td>
<td>Directed</td>
</tr>
<tr>
<td>---------------------</td>
<td>---------------------------------------------------------------------------</td>
</tr>
<tr>
<td>reachable</td>
<td>weakly connected; unilaterally connected; strongly connected; recursively connected;</td>
</tr>
<tr>
<td>connectivity</td>
<td>one of four kinds of connectivity above</td>
</tr>
<tr>
<td>geodesics/distance</td>
<td>sometimes, $d(n_i, n_j) \neq d(n_j, n_i)$</td>
</tr>
<tr>
<td>diameter</td>
<td>strongly connected or recursively connected graphs: the longest geodesic; unilaterally or weakly connected graphs: undefined.</td>
</tr>
<tr>
<td>complement</td>
<td>complement: same node set, complement arc set; converse: same node set, direction reversed arc set.</td>
</tr>
<tr>
<td>not available</td>
<td>tournament: a relation indicating superior performances or “beats” in competition.</td>
</tr>
</tbody>
</table>
Signed Graphs
Cycle | Sign of cycle
--- | ---
\( n_1 n_2 n_5 n_1 \) | \(- \times - \times - = -\)
\( n_3 n_4 n_1 n_3 \) | \(+ \times + \times + = +\)
\( n_1 n_2 n_3 n_1 \) | \(- \times - \times + = +\)

Example of a signed graph
<table>
<thead>
<tr>
<th>Graph</th>
<th>Signed graph</th>
<th>Signed directed graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>dyad</td>
<td>three possible states: 1) positive line; 2) negative line; 3) no line.</td>
<td>with additional direction information</td>
</tr>
<tr>
<td>triad</td>
<td>four possible states: zero, one, two, or three positive(negative) lines are present among the three nodes</td>
<td>with additional direction information</td>
</tr>
<tr>
<td>cycle</td>
<td>the sign of a cycle: the product of the signs of the lines included in the cycle</td>
<td>semicycle: a cycle in which the arcs may point in either direction</td>
</tr>
</tbody>
</table>
Valued Graphs
Valued Graphs & Valued Directed Graphs

To represent:
Valued relations
(e.g., the frequency of interaction among pair of people)

Possible values:
Integers $\rightarrow$ integer weighted digraph
Probabilities $\rightarrow$ Markov chains
(transition/stochastic matrices)
Non-numerical $\rightarrow$ network

Example of a valued directed graph (P.142)
<table>
<thead>
<tr>
<th>Graphs</th>
<th>Valued Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>nodal degree</td>
<td>average the values over all lines/arcs</td>
</tr>
<tr>
<td>dyad</td>
<td>valued directed graphs: two different values, usually to compare these two values</td>
</tr>
<tr>
<td>density</td>
<td>$\Delta = \sum V_k / g(g-1)$</td>
</tr>
<tr>
<td>path</td>
<td>value of a path: $\min(v_1, v_2, ..., v_k)$, a path at level $c$ reachability: two nodes are reachable at level $c$</td>
</tr>
<tr>
<td>path length</td>
<td>$\sum V_k$</td>
</tr>
</tbody>
</table>
Graph Generalization

• Multigraph
  – A simple graph or digraph that allows more than one set of lines

\[ \mathcal{N} = \{n_1, n_2, \ldots, n_g\} \quad \mathcal{L}^+ = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_R\} \]

• Hypergraph
  – To represent an affiliation network/a two-mode network (a set of actors + a set of events)
Example of a hypergraph (p.147)

\[ \mathcal{H} = (\mathcal{A}, \mathcal{B}) \]

point set \( \mathcal{A} = \{a_1, a_2, a_3, a_4\} \)

edge set \( \mathcal{B} = \{B_1, B_2, B_3\} \)

\[ B_1 = \{a_1, a_2\} \]
\[ B_2 = \{a_1, a_4\} \]
\[ B_3 = \{a_2, a_3, a_4\} \]

\[ \mathcal{H}^* = (\mathcal{B}, \mathcal{A}) \]

\[ A_1 = \{b_1, b_2\} \]
\[ A_2 = \{b_1, b_3\} \]
\[ A_3 = \{b_3\} \]
\[ A_4 = \{b_2, b_3\} \]
Relations
Mathematical Relations

\[ \mathcal{N} = \{n_1, n_2, \ldots, n_g\} \]

A set

\[ \mathcal{N} \times \mathcal{N} \]

Cartesian product of sets N and N, the collection of all ordered pairs of actors for whom substantive ties could be present.

A relation, \( R \), on the set \( \mathcal{N} \)

A subset of the Cartesian product \( \mathcal{N} \times \mathcal{N} \), all ordered pairs \( <n_i, n_j> \) for whom the substantive tie from \( i \) to \( j \) is present.

\[ iRj \quad < n_i, n_j > \in R \]
Properties of Relations

• Reflexivity
  – reflexive: all possible $<n_i, n_i>$ ties are present in R;
  – irreflexive: no $iR_i$
  – is not reflexive: neither reflexive nor irreflexive

• Symmetry
  – symmetric: $<n_i, n_j> \in R$ if and only if $<n_j, n_i> \in R$ (all dyads are mutual or null)
  – antisymmetric: $iR_j$ implies that not $jR_i$ (beats relation/tournament)
  – asymmetric: neither symmetric nor antisymmetric

• Transitivity
  – transitive: if whenever $iR_j$ and $jR_k$, then $iR_k$ (a friend of a friend is a friend)
Matrices
Sociomatrix (adjacency matrix): $X$

![Diagram of a sociomatrix showing relationships among six individuals with names: Allison, Sarah, Ross, Keith, Drew, and Eliot.]

Table 4.2. Example of a sociomatrix: “lives near” relation for six children

$$
\begin{array}{ccccccc}
   & n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\
\hline
n_1 & \cdot & 0 & 0 & 0 & 1 & 1 \\
n_2 & 0 & \cdot & 1 & 0 & 0 & 0 \\
n_3 & 0 & 1 & \cdot & 0 & 0 & 0 \\
n_4 & 0 & 0 & 0 & \cdot & 1 & 1 \\
n_5 & 1 & 0 & 0 & 1 & \cdot & 1 \\
n_6 & 1 & 0 & 0 & 1 & 1 & \cdot \\
\end{array}
$$
Table 4.3. Example of an incidence matrix: “lives near” relation for six children

<table>
<thead>
<tr>
<th></th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$l_4$</th>
<th>$l_5$</th>
<th>$l_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$n_5$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$n_6$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Matrices for Digraphs

Table 4.4. Example of a sociomatrix for a directed graph: friendship at the beginning of the year for six children

<table>
<thead>
<tr>
<th>Actor</th>
<th>Likes at beginning of year</th>
</tr>
</thead>
<tbody>
<tr>
<td>n₁</td>
<td>Allison</td>
</tr>
<tr>
<td>n₂</td>
<td>Drew</td>
</tr>
<tr>
<td>n₃</td>
<td>Eliot</td>
</tr>
<tr>
<td>n₄</td>
<td>Keith</td>
</tr>
<tr>
<td>n₅</td>
<td>Ross</td>
</tr>
<tr>
<td>n₆</td>
<td>Sarah</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>n₁</th>
<th>n₂</th>
<th>n₃</th>
<th>n₄</th>
<th>n₅</th>
<th>n₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>n₁</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>n₂</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>n₃</td>
<td>0</td>
<td>1</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>n₄</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>n₅</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>n₆</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Diagram of the digraph:

- Allison → Drew
- Sarah → Drew
- Ross → Eliot
- Keith → Eliot

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Matrices for More Graphs

• Matrices for Valued Graphs: X
  – $X_{ij}$: the value associated with the line/arc

• Matrices for Two-Mode Networks
  – $N=\{n_1,n_2,\ldots,n_g\}$, $M=\{m_1,m_2,\ldots,m_h\}$: a $g \times h$ matrix

• Matrices for Hypergraphs: $A$
  – incidence matrix
  – point set $N=\{n_1,n_2,\ldots,n_g\}$, edge set
    $M=\{m_1,m_2,\ldots,m_h\} \Rightarrow A=\{a_{ij}\}$, $a_{ij}=1$ if point $n_i$ is in edge $M_j$
Basic Matrix Operations
Permutation

• Rearrangement of rows, and simultaneously of columns

Example of matrix permutation (p.156)
Transpose

- Interchanging the rows and columns of the original matrix
  - a tie from row actor $i$ to column actor $j$ → row actor $i$ received a tie from column actor $j$

Example of matrix transpose (p.157)
• **Addition and Subtraction**

  $X,Y$: both of size $g$ by $h$

  $Z = X + Y$ where $z_{ij} = x_{ij} + y_{ij}$; $Z = X - Y$ where $z_{ij} = x_{ij} - y_{ij}$

• **Matrix Multiplication**

  – used to study walks and reachability

  $Y$: size of $g$ by $h$; $W$: size of $h$ by $k$

  $Z = YW$ where

  $z_{ij} = \sum_{l=1}^{h} y_{il} w_{lj}$

  ![Matrix Multiplication Example](image)

  $z_{11} = (1 \times 0) + (0 \times 1) + (1 \times 2) = 0 + 0 + 2 = 2$

  $z_{12} = (1 \times 2) + (0 \times 1) + (1 \times 3) = 2 + 0 + 3 = 5$

  $z_{21} = (1 \times 0) + (3 \times 1) + (2 \times 2) = 0 + 3 + 4 = 7$

  $z_{22} = (1 \times 2) + (3 \times 1) + (2 \times 3) = 2 + 3 + 6 = 11$
• Powers of a Matrix
  \( X^p \) (X to the \( p \)th power):
  the matrix product of X times itself, \( p \) times

• Boolean Matrix Multiplication
  – used to study walks and reachability

\[
Z^\otimes = X \otimes Y
\]

\[
z_{ij}^\otimes = \begin{cases} 
1 & \text{if } \sum_{l=1}^{h} y_{il} w_{lj} > 0 \\
0 & \text{if } \sum_{l=1}^{h} y_{il} w_{lj} = 0.
\end{cases}
\]
Computing Simple Network Properties
Walks and Reachability

\[ X^2 \]

\[ x_{ij} = \sum_{k=1}^{g} x_{ik} x_{kj} \]

The number of walks of length 2 between nodes \( n_i \) and \( n_j \), for all \( k \).

\[ X^p \]

The number of walks of length \( P \) between nodes \( n_i \) and \( n_j \), for all \( k \).

\[ X^{[\Sigma]} = X + X^2 + X^3 + \ldots + X^{g-1} \]

The total number of walks from \( n_i \) to \( n_j \), of any length less than or equal to \( g-1 \)

Pairs of nodes that are not reachable : a 0 in cell \((i, j)\)
Geodesic and Distance

• (Geodesic) Distance
  – the length of a shortest path between two notes:
  – the first power p for which the (i, j) element is non-zero: \( d(i,j) = \min_p X_{ij}[^p] > 0 \)

• Diameter
  – the largest geodesic in the graph or digraph
Powers of a sociomatrix for a directed graph (p.162)

<table>
<thead>
<tr>
<th></th>
<th>n₁</th>
<th>n₂</th>
<th>n₃</th>
<th>n₄</th>
<th>n₅</th>
<th>n₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>X³</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>n₁</td>
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<td>0</td>
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<tr>
<td>X²</td>
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\[ X^4 = \begin{array}{cccccc|ccc} n₁ & n₂ & n₃ & n₄ & n₅ & n₆ & n₁ & n₂ & n₃ & n₄ & n₅ & n₆ \\ \hline n₁ & 0 & 0 & 3 & 0 & 0 & 3 \\ n₂ & 0 & 4 & 0 & 0 & 0 & 0 \\ n₃ & 0 & 0 & 2 & 0 & 0 & 2 \\ n₄ & 0 & 0 & 1 & 0 & 0 & 1 \\ n₅ & 0 & 2 & 0 & 0 & 0 & 0 \\ n₆ & 0 & 0 & 2 & 0 & 0 & 2 \end{array} \]

\[ X^5 = \begin{array}{cccccc|ccc} n₁ & n₂ & n₃ & n₄ & n₅ & n₆ & n₁ & n₂ & n₃ & n₄ & n₅ & n₆ \\ \hline n₁ & 0 & 6 & 0 & 0 & 0 & 0 \\ n₂ & 0 & 0 & 4 & 0 & 0 & 4 \\ n₃ & 0 & 4 & 0 & 0 & 0 & 0 \\ n₄ & 0 & 2 & 0 & 0 & 0 & 0 \\ n₅ & 0 & 0 & 2 & 0 & 0 & 2 \\ n₆ & 0 & 4 & 0 & 0 & 0 & 0 \end{array} \]
Computing Nodal Degrees and Density

- Nodal degree

\[ d(n_i) = \sum_{j=1}^{L} I_{ij} \]
\[ d(n_i) = \sum_{j=1}^{g} x_{ij} = \sum_{i=1}^{g} x_{ij} = x_{i+} = x_{+j} \]

- Directed Graph

\[ d_0(n_i) = \sum_{j=1}^{g} x_{ij} = x_{i+} \]
\[ d_f(n_i) = \sum_{j=1}^{g} x_{ji} = x_{+i} \]

- Density

\[ \Delta = \frac{\sum_{i=1}^{g} \sum_{j=1}^{g} x_{ij}}{g(g-1)} \]
Properties of Graphs, Relations, and Matrices

• Reflexivity
  – reflexive: $x_{ii} = 1$
  – irreflexive: $x_{ii}$ undefined
  – is not reflexive: has some, but not all, values of $x_{ii} = 1$

• Symmetry
  – $x_{ij} = x_{ji}$

• Transitivity
  – if there is a walk $n_i \rightarrow n_k \rightarrow n_j$ for at least one node $n_k$: $X_{ij}^{[2]} \geq 1$
  – transitive: whenever $X_{ij}^{[2]} \geq 1$, then $x_{ij}$ must equal 1.