Deep Learning in Wireless Communications

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Introduction

Motivation

- Several Magazines, Journals, Conferences
- IEEE ComSoc Technical Committee
  - Emerging Technologies Initiative Machine Learning for Communications

Materials

- Deep Learning
  Ian Goodfellow, Yoshua Bengio and Aaron Courville
  https://www.deeplearningbook.org/
- Dive into Deep Learning
  Aston Zhang, Zachary Lipton, Mu Li and Alexander Smola
  https://d2l.ai
- Machine Learning: A Probabilistic Perspective
  Kevin P. Murphy
  https://probml.github.io/pml-book/
- Several published papers

Wireless Networking Applications (Some use cases)

- Channel Modeling
- Channel Estimation
- Beamforming and beam prediction
- Antenna tilting
- RF fingerprinting
- Spectrum availability prediction
- Modulation detection
- Waveform generation
- Channel Coding

- Resource Allocation
- Path planning for autonomous systems
- Handover
- Wireless user behavior
- Wireless content prediction
- UAV trajectory prediction

Why Deep Learning can yield better results in Wireless Communication?

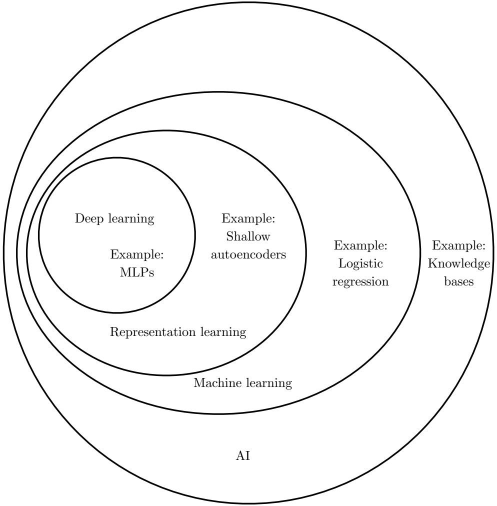
- Signal processing in Tx-Rx chains have been developed (and are optimal) for Gaussian channels
- Often, it is difficult to find a closed form representation of a problem
- Computational complexity of optimal solutions might be high
- New areas of research

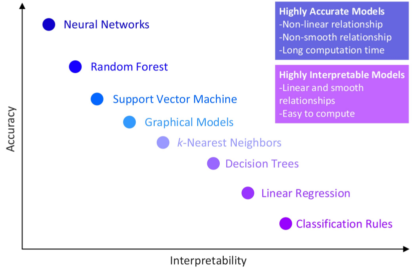
Understanding Wireless Data

- Signals are complex valued (I, Q), whereas image data is three dimensional (RGB)
- Spectrogram can be considered images, but we lose information
- Range varies from [0-254] in image, whereas between [-1,+1] in wireless signals

Challenge: Dataset

Evolution of Deep Learning

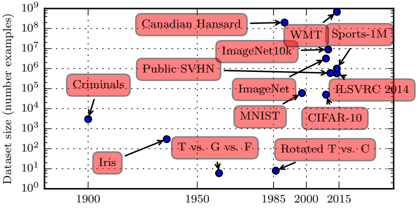


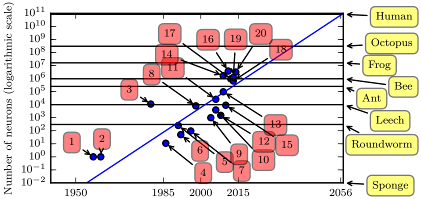


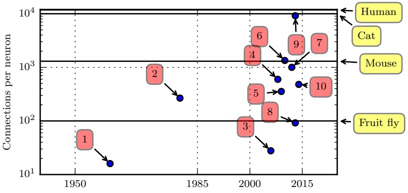
M. E. Morocho-Cayamcela, H. Lee and W. Lim, "Machine Learning for 5G/B5G Mobile and Wireless Communications: Potential, Limitations, and Future Directions," in IEEE Access, vol. 7, pp. 137184-137206, 2019, doi: 10.1109/ACCESS.2019.2942390.

Key Reasons for Success of Deep Learning

- Increasing Dataset Sizes
  - 5000 labeled samples/category
- Increasing Model Sizes
  - Hidden layers doubled every 2.4years
  - Availability of faster CPUs
  - Advent of GPUs
  - Faster network connectivity
  - Better software infrastructure

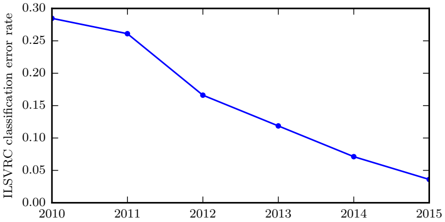






Improved Accuracy

Decade	Dataset	Memory	Floating point calculations per second
1970	100 (Iris)	1 KB	100 KF (Intel 8080)
1980	1 K (House prices in Boston)	100 KB	1 MF (Intel 80186)
1990	10 K (optical character recognition)	10 MB	10 MF (Intel 80486)
2000	10 M (web pages)	100 MB	1 GF (Intel Core)
2010	10 G (advertising)	1 GB	1 TF (Nvidia C2050)
2020	1 T (social network)	100 GB	1 PF (Nvidia DGX-2)



Learning Model

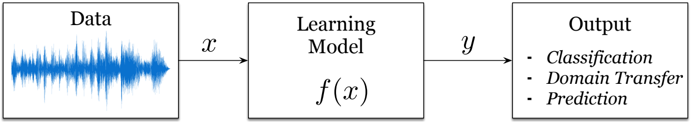


Figure: Approximate $f(x)$ from the data

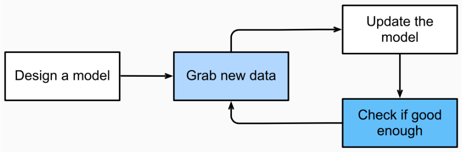


Figure: A typical training process

- *Learning* is the process by which we discover the right setting of the knobs yielding the desired behavior from our model. In other words, we *train* our model with data.
- Wireless Applications: modulation detection, RF fingerprinting, channel estimation, channel modeling, generate waveforms

Key Components of Learning

- The data that we can learn from: constitutes attributes or *features* from which the model should learn.
  - Training Set: set of examples used to fit the parameters of the model
  - Validation/Testing Set: set of examples used to test the performance of the model after training

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  - a mathematical formulation to measure performance in each iteration (epoch) of training
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- An algorithm to adjust the model's parameters to optimize the objective function.

Types of Machine Learning Problems

- Supervised Learning: addresses the task of predicting labels given input features (labels)
  - Example Modulation Detection: Labels provided during training (modulation order)
  - Types: Regression, Classification, Tagging, Search, Recommender Systems, Sequence Learning

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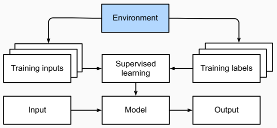

Figure: Interacting with an Environment

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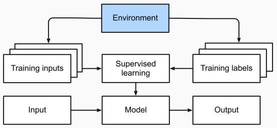
- Reinforcement Learning: develop an agent that interacts with an environment, takes actions, a policy to reward the action



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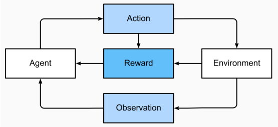

Figure: Reinforcement Learning

Machine Learning Basics

Machine Learning Basics

- Linear Algebra
- Probability
- Calculus

Scalars and Vectors

- A scalar is a single number
  - Integers, real numbers, rational numbers, etc.
  - We denote it with italic font: a, n, x
- A vector is a 1-D array of numbers
  - Can be real, binary, integer, etc.
  - Example notation for type and size: $\boldsymbol{x} \in \mathbb{R}^n$
- A matrix is a 2-D array of numbers
  - Can be real, binary, integer, etc.
  - Example notation for type and size: $\boldsymbol{A} \in \mathbb{R}^{m \times n}$
- A tensor is an array of numbers, that may have
  - zero dimensions, and be a scalar
  - one dimension, and be a vector
  - two dimensions, and be a matrix
  - or more dimensions.

Vector:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix Operations

- Matrix Transpose: $\mathbf{B} = \mathbf{A}^\top$

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- Hadamard Product: Elementwise multiplication $C = \mathbf{A} \odot \mathbf{B}$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \cdots & a_{mn}b_{mn} \end{bmatrix}$$

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- Reduction: Sum of the elements

$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

Matrix Operations

- Matrix Multiplication: $\mathbf{C} = \mathbf{AB}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix}, \quad \mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_n^\top \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \mathbf{a}_1^\top \mathbf{b}_2 & \cdots & \mathbf{a}_1^\top \mathbf{b}_m \\ \mathbf{a}_2^\top \mathbf{b}_1 & \mathbf{a}_2^\top \mathbf{b}_2 & \cdots & \mathbf{a}_2^\top \mathbf{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^\top \mathbf{b}_1 & \mathbf{a}_n^\top \mathbf{b}_2 & \cdots & \mathbf{a}_n^\top \mathbf{b}_m \end{bmatrix}$$

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- Matrix Inversion: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$, where $\mathbf{I}$ is the identity matrix
  Example: Let $\mathbf{Ax} = \mathbf{b}$ be a system of linear equation.
  Then, $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$, which implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Matrix Operations

- Norm: Function to measure size of a vector
- Does not represent dimensionality but rather the magnitude of the components.

Matrix Operations

- Norm: Function to measure size of a vector
- Does not represent dimensionality but rather the magnitude of the components.
- A vector norm is a function f that maps a vector to a scalar, satisfying following properties:
  1. if all elements of a vector are scaled by a constant factor α, its norm also scales by α
     $f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$
  2. Triangle inequality
     $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$
  3. Norm must be non-negative
     $f(\mathbf{x}) \geq 0$
  4. smallest norm is achieved by a vector consisting of all zeros
     $\forall i, [\mathbf{x}]_i = 0 \Leftrightarrow f(\mathbf{x}) = 0$

Matrix Operations

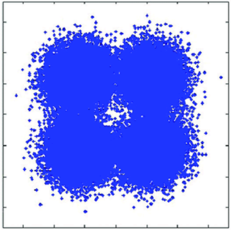
Norms

- Generalized Form: (L_p norm) $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, where $p \in \mathbb{R}, p \geq 1$
- L_2 norm is the Eucledian distance
  Suppose elements in the n-dimensional vector $\mathbf{x}$ are $x_1, \ldots, x_n$
  Then, L_2 norm of $\mathbf{x}$ is $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$,
- L_1 norm is $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$
  Compared to L_2 norm, it is less influenced by outliers
- Max norm (L_∞): absolute value of the element with the largest magnitude in the vector
  $\|\boldsymbol{x}\|_\infty = \max_i |x_i|$
- Frobenius norm of matrices is similar to L_2 Norm of vectors
  $\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2}$, where $\mathbf{X} \in \mathbb{R}^{m \times n}$

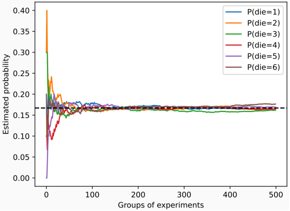
Objective functions are described as norms: minimize distance between predictions and ground-truth observations

Probability: Connection to Machine Learning

- Machine Learning is probabilistic (not deterministic)



(a) 95% QPSK



(b) Law of large numbers

- Basic Probability Theory: *law of large numbers*
- Sources of uncertainty
  - Inherent stochasticity in the system being modeled
  - Incomplete observability
  - Incomplete modeling

Basic Probability

- Sample space or outcome space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, where output of random experiment is an event $\mathcal{A}$.
- Probability of an event $\mathcal{A}$ in the given sample space $\mathcal{S}$, denoted as $P(\mathcal{A})$ satisfies the following properties:
  - For any event $\mathcal{A}$, its probability is never negative, i.e., $P(\mathcal{A}) \geq 0$
  - Probability of the entire sample space is 1, or $P(\mathcal{S}) = 1$
  - For any countable sequence of events, $\mathcal{A}_1, \mathcal{A}_2, \ldots$ that are *mutually exclusive*, ($\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for all $i \neq j$) the probability that any happens is equal to the sum of their individual probabilities, or $P(\bigcup_{i=1}^{\infty} \mathcal{A}_i) = \sum_{i=1}^{\infty} P(\mathcal{A}_i)$

Random Variables: A random variable is a variable that can take on different values randomly. Random variables may be discrete (integer or states) or continuous (real numbers).

Probability Distributions

- A *probability distribution* is a description of how likely a random variable or set of random variables is to take on each of its possible states.
- The probability that $\mathrm{x} = x$ is denoted as $P(x)$
- *Probability mass function (PMF)*: A probability distribution over discrete variables
- *Probability density function (PDF)*: A probability distribution over continuous variables

Multiple Random Variables

- Joint Probability: $P(A = a, B = b)$, where $P(A = a, B = b) \leq P(A = a)$
- Conditional Probability: $P(B = b \mid A = a)$ is the ratio $0 \leq \frac{P(A=a,B=b)}{P(A=a)} \leq 1$
  - probability of $B = b$, provided that $A = a$ has occurred
  - used in causal modeling

Bayes' Theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- The probability of two events A and B happening is $P(A \cap B) = P(A)P(B \mid A)$
- Similarly, $P(B \cap A) = P(B)P(A \mid B)$
- Equating them, $P(B)P(A \mid B) = P(A)P(B \mid A)$
- Hence, $P(A \mid B) = \frac{P(B|A)P(A)}{P(B)}$

Expectation, Variance and Covariance

To summarize key characteristics of probability distributions, we need some measures.

- The *expectation*, or expected value, of some function $f(x)$ with respect to a probability distribution $P(x)$ is the average, or mean value, that f takes on when x is drawn from P

$$\mathbb{E}_{x \sim P}[f(x)] = \sum_x f(x)P(x)$$

- The *variance* gives a measure of how much the values of a function of a random variable x vary as we sample different values of x from its probability distribution

$$\text{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- *Covariance* describes how much two values are linearly related to each other

$$\text{Cov}(f(x), g(y)) = \mathbb{E}[(f(x) - \mathbb{E}[f(x)])(g(y) - \mathbb{E}[g(y)])]$$

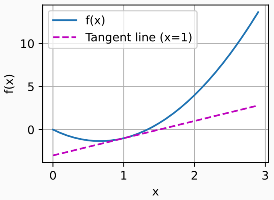
Differential Calculus

Optimization in neural networks uses *Differential Calculus*

- If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has scalar input and output
- *Derivative* of f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- if $f'(a)$ exists, f is said to be differentiable at a
- The derivative $f'(x)$ is instantaneous rate of change of $f(x)$ with respect to x.
- Common notations: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$
- Example: $f(x) = x^2 - 3x, f'(x) = 2x - 3$ is the tangent



Rules of Differentiation

- Constant Multiple Rule

$$\frac{d}{dx}[Cf(x)] = C\frac{d}{dx}f(x)$$

- Sum Rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

- Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

- Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Partial Derivatives and Gradients

- In deep learning, functions depend on many variables
- In a multivariate function, $y = f(x_1, x_2, \ldots, x_n)$

$$\frac{\partial y}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

- Notations: $\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f = D_{x_i} f$
- The *gradient vector* of a multivariate function is concatenated partial derivatives of the function with respect to all its variables, the gradient is
- If $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$ is a vector,

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \nabla f(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n}\right]^\top$$

Chain Rule

- Multivariate functions in deep learning are composite
- Chain rule enables us to differentiate composite functions
- If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

- For arbitrary number of variables, (y has variables $u_1, u_2, \ldots, u_m$ and x has variable $x_1, x_2, \ldots, n_m$)

$$\frac{dy}{dx_i} = \frac{dy}{du_1}\frac{du_1}{dx_i} + \frac{dy}{du_2}\frac{du_2}{dx_i} + \cdots + \frac{dy}{du_m}\frac{du_m}{dx_i}$$

Linear Neural Networks

Linear Regression (Statistics)

- *Regression*: A set of methods for modeling the relationship between one or more independent variables and a dependent variable
- Assumptions:
  - Relationship between the independent variables x and the dependent variable y is linear
  - Noise is Gaussian
- Example: House price depends on *features* (age and area)

$$\text{price} = w_{\text{area}} \cdot \text{area} + w_{\text{age}} \cdot \text{age} + b, \quad \text{where } w_{area} \,\&\, w_{age} \text{ are } \textit{weights} \text{ and } b \text{ is } \textit{bias}$$

- For d features, the prediction, $\hat{y} = w_1 x_1 + \ldots + w_d x_d + b$
- In linear algebra notations,

$$\hat{y} = \mathbf{w}^\top \mathbf{x} + b, \text{where features of single data examples } \mathbf{x} \in \mathbb{R}^d \text{ and weights } \mathbf{w} \in \mathbb{R}^d$$

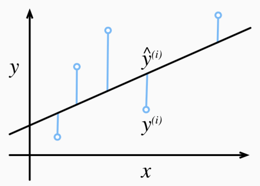
- For a collection of features $\mathbf{X}$, the predictions $\hat{\mathbf{y}} \in \mathbb{R}^n$ can be expressed via the matrix-vector product

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{w} + b$$

Loss Function

- Measure of fitness: quantifies the distance between the real and predicted value of the target
- Usually a non-negative number, smaller is better, 0 for perfect prediction
- Most popular loss function in regression problems is the squared error

$$l^{(i)}(\mathbf{w}, b) = \frac{1}{2}\left(\hat{y}^{(i)} - y^{(i)}\right)^2$$



- Mean Squared Error (MSE): Average of losses on entire dataset quantifies quality of a model

$$L(\mathbf{w}, b) = \frac{1}{n}\sum_{i=1}^{n} l^{(i)}(\mathbf{w}, b) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2}\left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - y^{(i)}\right)^2$$

- Objectives during training: $\mathbf{w}^*, b^* = \operatorname{argmin}_{\mathbf{w},b}\ L(\mathbf{w}, b)$

Analytic Solution

- Linear regression is a simple optimization problem
- Steps to solve:
  – Subsume the bias b into the parameter $\mathbf{w}$ by appending a column to the design matrix consisting of all ones
  – Prediction problem is to minimize $\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2$
  – Taking the derivative of the loss w.r.t. $\mathbf{w}$ and setting it to 0, yields closed form solution

$$\mathbf{w}^* = (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{y}$$

- *Linear regression is extremely simple and limited learning algorithm*

Gradient Descent as Optimization Algorithm

- Optimization refers to the task of either minimizing or maximizing some function $f(x)$ by altering x
- In deep learning, most popular optimization is done by *Gradient Descent* [Cauchy, 1847]
  – Reduce the error by updating the parameters in the direction that iteratively lowers the loss function
  – Requires taking the derivative of the loss function, which is an average of the losses computed on *entire dataset*
  – Extremely slow in practice
- *Minibatch Stochastic Gradient Descent*: sample a random *minibatch* of examples every time there is a need to compute the update

Minibatch Stochastic Gradient Descent

- Randomly sample a minibatch $\mathcal{B}$ consisting of a fixed number of training examples
- Compute derivative (gradient) of average loss on minibatch with regard to model parameters
- Multiply the gradient by a predetermined positive value η
- Subtract the resulting term from the current parameter values

$$(\mathbf{w}, b) \leftarrow (\mathbf{w}, b) - \frac{\eta}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \partial_{(\mathbf{w}, b)} l^{(i)}(\mathbf{w}, b), \text{ where } |\mathcal{B}| \text{ is number of examples in each minibatch \& } \eta \text{ is learning rate}$$

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- Steps of the algorithm:
  (i) initialize the values of the model parameters, typically randomly
  (ii) iteratively sample random minibatches from the data, updating the parameters in the direction of the negative gradient

$$\mathbf{w} \leftarrow \mathbf{w} - \frac{\eta}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \partial_{\mathbf{w}} l^{(i)}(\mathbf{w}, b) = \mathbf{w} - \frac{\eta}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \mathbf{x}^{(i)} \left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - y^{(i)}\right),$$

$$b \leftarrow b - \frac{\eta}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \partial_b l^{(i)}(\mathbf{w}, b) = b - \frac{\eta}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - y^{(i)}\right).$$

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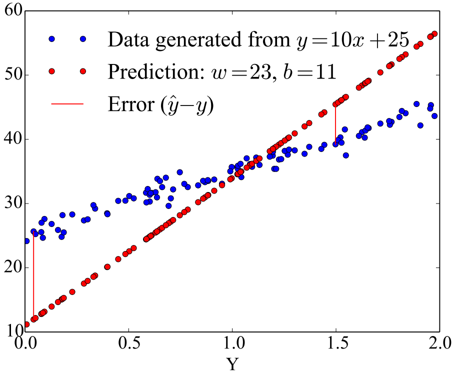
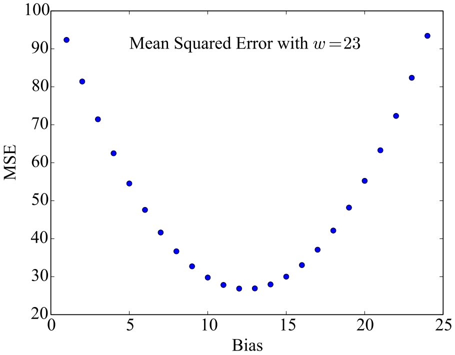
- The values of $|\mathcal{B}|$ and η are manually pre-specified and not learned through model training

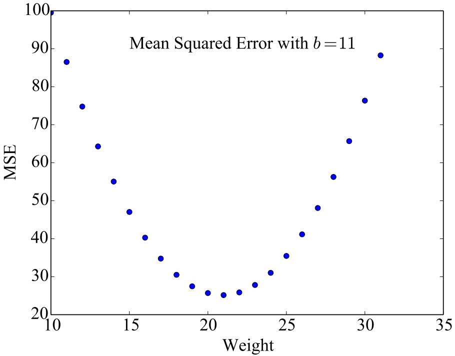
Linear Regression Example



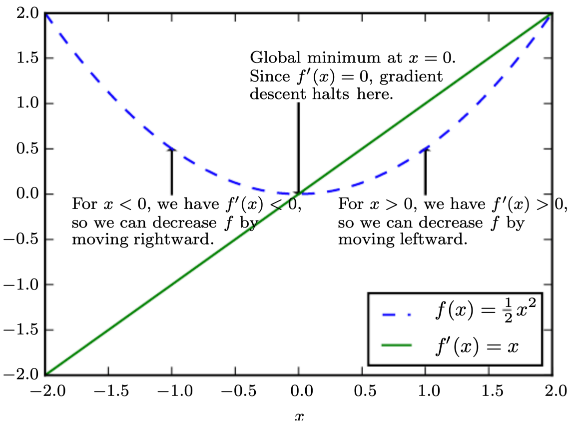

```
#Generate Data (blue dots)
X = 2 * np.random.rand(100,1)
Y = 25 + 10 * X+np.random.randn(100,1)


#Random Prediction (red dots)
weight = 23.0
bias = 11.0
Y_hat = bias + weight * X


#Error (red lines)
error = Y_hat - Y
sq_err = 0.5 * pow((Y - Y_hat),2)
mse = np.mean(sq_err)
```

The Gradient

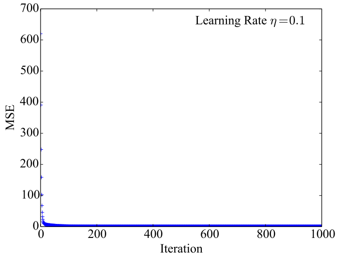


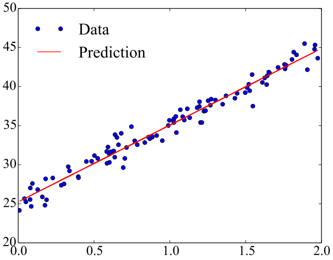


Gradient Descent



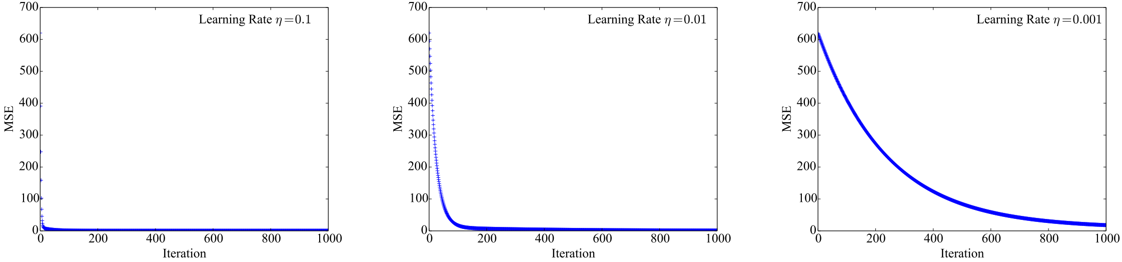
Calculate Gradient and Update






```
for i in range(1000):
    #Calculate the gradient
    err = Y-Y_hat
    grad_bias = -(err)
    grad_weight = -(err)*X

    #Mean of the gradients
    g_w = np.mean(grad_weight)
    g_b = np.mean(grad_bias)

    #Update the weight and bias
    weight = weight - rate * g_w
    bias = bias - rate * g_b
```

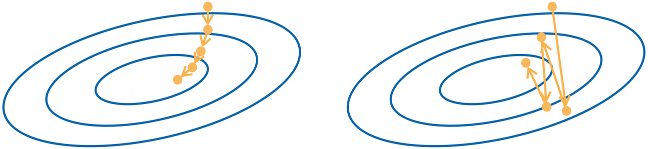
The Learning Rate



- *Learning Rate* and *Batch Size* are tunable but not updated in the training loop
- They are called *Hyperparameters*
- Hyperparameter tuning is the process by which hyperparameters are chosen
- Hyperparameters are adjusted based on the results of the training loop

Choice of Hyperparameters

- Learning Rate:
  - Too small will be too slow, too large might oscillate

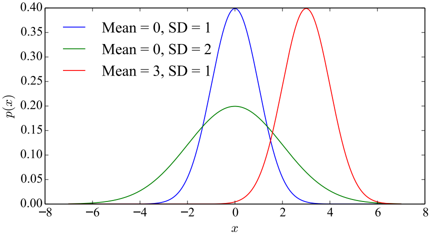


- Batch Size:
  - Too small: Workload is too small, hard to fully utilize computation resources
  - Too large: Memory issues, Wasteful computation when x_i are identical

Motivation behind Squared Loss

- Probability Density Function (PDF) of a normal distribution with mean μ and variance σ^2

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$



- Changing the mean corresponds to a shift along the X-axis
- Increasing the variance spreads distribution out, lowering its peak

Motivation behind Squared Loss

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$$P(y \mid \mathbf{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y - \mathbf{w}^\top \mathbf{x} - b)^2\right)$$

- According to the principle of *maximum likelihood*, the best values of parameters **w** and **b** are those that maximize the likelihood of the entire dataset $P(\mathbf{y} \mid \mathbf{X}) = \prod_{i=1}^{n} p(y^{(i)}|\mathbf{x}^{(i)})$

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- *Maximum Likelihood Estimators*: Estimators chosen according to the principle of maximum likelihood
- Maximizing product of many exponential functions could be a hard problem
- Instead we can choose maximizing the log of the likelihood that does not change the objective
- Convert to minimization by taking *negative log-likelihood*

$$-\log P(\mathbf{y} \mid \mathbf{X}) = \sum_{i=1}^{n} \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)} - b\right)^2$$

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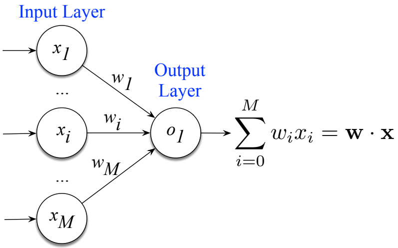
- Assume σ is a constant, it is identical to squared loss

Single Layer Neural Network (NN): Linear Regression

- Input Layer: Number of inputs = M = Feature Dimensionality, given
- Output Layer: Number of outputs = 1
- One single computed neuron
- Number of layers of the NN = 1 [input layer is not counted]
- Linear Regression can be cast in this NN
- *Fully-connected layer* or *Dense layer*: when all inputs are connected to all outputs



Connection to Neurons

- Information x_i arriving from other neurons (or environmental sensors such as the retina) is received in the dendrites.
- That information is weighted by synaptic weights w_i determining the effect of the inputs (e.g., activation or inhibition via the product $x_i w_i$).
- The weighted inputs arriving from multiple sources are aggregated in the nucleus as a weighted sum $y = \sum_i x_i w_i + b$.
- This information is then sent for further processing in the axon y, after some nonlinear processing via $\sigma(y)$.
- From there it either reaches its destination (e.g., a muscle) or is fed into another neuron via its dendrites.

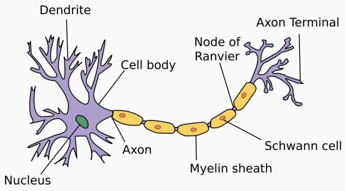
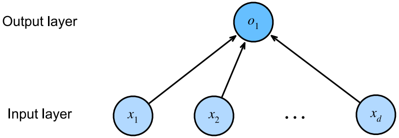


Figure: Dendrites (input terminals), the nucleus (CPU), the axon (output wire), and the axon terminals (output terminals), enables connections to other neurons via synapses

Regression vs Classification

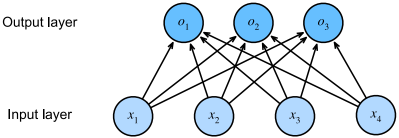
Regression: estimates a continuous value

- How much?
- Example Application: Channel estimation
- Single continuous output
- Natural scale in $\mathbb{R}$
- Loss in terms of difference: $y - f(x)$



Classification: predicts a discrete category

- Which one?
- Example Application: Modulation recognition, RF fingerprinting
- Multiple classes, typically multiple outputs
- Score should reflect confidence



Multi-class Classification

- *One-hot Encoding*: A one-hot encoding is a vector with as many components as the number of categories
- The component corresponding to particular instance's category is set to 1 and all other components are set to 0
- Example: QPSK has 4 constellation points

Constellation	**Hot-one Encoded Labels**
1+j	(1, 0, 0, 0)
-1+j	(0, 1, 0, 0)
-1-j	(0, 0, 1, 0)
1-j	(0, 0, 0, 1)

Network Architecture

- Estimate the conditional probabilities associated with all the possible classes
- A model with multiple outputs, one per class
- With linear models, we need as many affine functions as we have outputs
- Each output will correspond to its own affine function
- Example: 4 features and 3 output categories
- 12 scalars to represent the weights, 3 scalars to represent the biases

$$
\begin{aligned}
o_1 &= x_1 w_{11} + x_2 w_{12} + x_3 w_{13} + x_4 w_{14} + b_1,\\
o_2 &= x_1 w_{21} + x_2 w_{22} + x_3 w_{23} + x_4 w_{24} + b_2,\\
o_3 &= x_1 w_{31} + x_2 w_{32} + x_3 w_{33} + x_4 w_{34} + b_3.
\end{aligned}
$$

$$\mathbf{o} = \mathbf{W}\mathbf{x} + \mathbf{b}$$

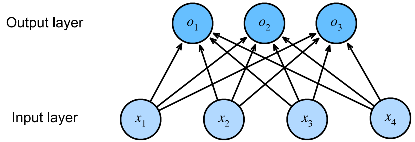


Figure: Softmax regression is a fully connected single-layer neural network

Softmax Operation

- Interpret the outputs of the model as probabilities
- Optimize parameters to produce probabilities that maximizes the likelihood of the observed data
- Need output $\hat{y}_j$, to be interpreted as the probability that a given item belongs to class j
- Choose the class with the largest output value as the prediction $\operatorname{argmax}_j y_j$
- For example, $\hat{y}_1 = 0.1$, $\hat{y}_2 = 0.8$ and $\hat{y}_3 = 0.1$ indicates output is Category 2.

Softmax Operation

- Goals:
  - Logits should not be negative
  - Sum of all logits equals 1
  - Model should be differentiable
- Softmax function:

$$\hat{\mathbf{y}} = \mathrm{softmax}(\mathbf{o}) \quad \text{where} \quad \hat{y}_j = \frac{\exp(o_j)}{\sum_k \exp(o_k)}.$$

- Exponential function, $\exp(o_j)$ ensures non negativity
- Dividing by their sum $\sum_k \exp(o_k)$ ensures they sum up to 1.
- Prediction:

$$\operatorname*{argmax}_j \hat{y}_j = \operatorname*{argmax}_j o_j.$$

Loss Function

- The softmax function outputs a vector $\hat{\mathbf{y}}$, which can be interpreted as estimated conditional probabilities of each class given any input $\mathbf{x}$
$$\hat{y} = P(y = \text{Category } 1 \mid \mathbf{x})$$
- For the entire dataset, $P(\mathbf{Y} \mid \mathbf{X}) = \prod_{i=1}^{n} P(\mathbf{y}^{(i)} \mid \mathbf{x}^{(i)})$
- According to maximum likelihood estimation, we maximize $P(\mathbf{Y} \mid \mathbf{X})$
- This is equivalent to minimizing the negative log-likelihood
$$-\log P(\mathbf{Y} \mid \mathbf{X}) = \sum_{i=1}^{n} -\log P(\mathbf{y}^{(i)} \mid \mathbf{x}^{(i)}) = \sum_{i=1}^{n} l(\mathbf{y}^{(i)}, \hat{\mathbf{y}}^{(i)})$$
- For any pair of label $\mathbf{y}$ and model prediction $\hat{\mathbf{y}}$ over q classes, the loss function l is
$$l(\mathbf{y}, \hat{\mathbf{y}}) = -\sum_{j=1}^{q} y_j \log \hat{y}_j \qquad \text{– \textit{Cross Entropy Loss}}$$
- $\mathbf{y}$ one-hot vector of length q, so sum over all its coordinates j vanishes for all except one term
- $\hat{y}_j$ are probabilities, so their logarithms are never greater than 0.

Softmax and Derivatives

Loss:

$$l(\mathbf{y}, \hat{\mathbf{y}}) = -\sum_{j=1}^{q} y_j \log \frac{\exp(o_j)}{\sum_{k=1}^{q} \exp(o_k)}$$

$$= \sum_{j=1}^{q} y_j \log \sum_{k=1}^{q} \exp(o_k) - \sum_{j=1}^{q} y_j o_j$$

$$= \log \sum_{k=1}^{q} \exp(o_k) - \sum_{j=1}^{q} y_j o_j.$$

Derivative:

$$\partial_{o_j} l(\mathbf{y}, \hat{\mathbf{y}}) = \frac{\exp(o_j)}{\sum_{k=1}^{q} \exp(o_k)} - y_j = \mathrm{softmax}(\mathbf{o})_j - y_j.$$

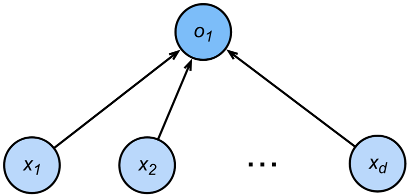
Gradient is the difference between the observation and estimate

Multi Layer Perceptron

Perceptron

- An algorithm for supervised learning of binary classifiers
- Binary classification outputs 0 or 1
  - vs. scalar real value for regression
  - vs. probabilities for logistic regression
- Given input **x**, weight **w** and bias b, perceptron outputs:

$$o = \sigma\left(\langle \mathbf{w}, \mathbf{x} \rangle + b\right) \qquad \sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$



Binary Classification

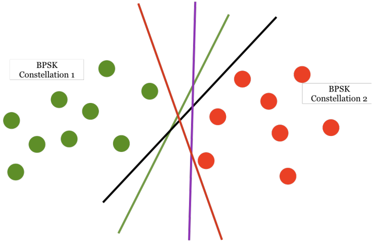


Figure: Example Application: Decoding BPSK

initialize $w = 0$ and $b = 0$
repeat
  if $y_i \left[\langle w, x_i \rangle + b\right] \leq 0$ **then**
    $w \leftarrow w + y_i x_i$ and $b \leftarrow b + y_i$
  end if
until all classified correctly

- Equals to SGD (batch size is 1) with the following loss

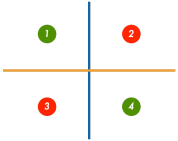
$$\ell(y, \mathbf{x}, \mathbf{w}) = \max(0, -y\langle \mathbf{w}, \mathbf{x} \rangle)$$

- Convergence Theorem: If a data set is linearly separable, the Perceptron will find a separating hyperplane in a finite number of updates.

Assumption of Linearity

Linearity is a strong assumption

- XOR Problem (Minsky Papert, 1969): A perceptron cannot learn an XOR function

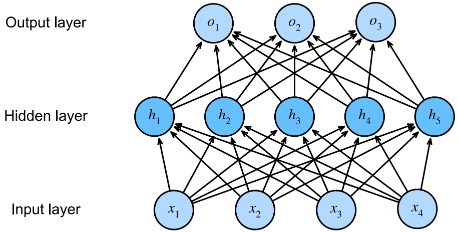


	1	2	3	4
	+	-	+	-
	+	+	-	-
product	+	-	-	+

- Wireless domain:
  – Frequency response of a multipath channel
  – Non-linear region of hardware

Multilayer Perceptrons or Feed Forward Networks

- Multiple layers of perceptrons
- The goal of a feedforward network is to approximate some function f^*
- Defines a mapping $y = f(x;\theta)$ and learns the value of the parameters θ that result in the best function approximation.
- These models are called feedforward
  – Information flows through the function being evaluated from x
  – Intermediate computations used to define f
  – Outputs y
  – No feedback connections

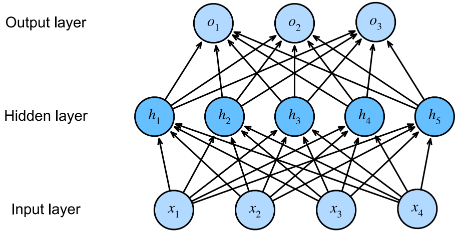


Feed Forward Network

- Called *networks* as they represent composition of many functions
- Model can be represented by Directed Acyclic Graph
- Functions connected in a chain: $f(\boldsymbol{x}) = f^{(3)}\left(f^{(2)}\left(f^{(1)}(\boldsymbol{x})\right)\right)$
- $f(i)$ is called the i^{th} layer of the network
- Length of chain is the *depth* of the network
  – It is a *Hyperparameter*
- Final layer is the *output* layer
- Layers in between input and output are *hidden* layers
  – training data does not show any output for each of these layers

Single Hidden Layer

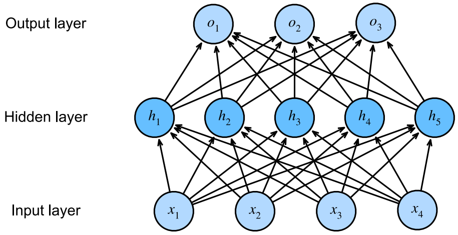
- Minibatch of n examples, each example has d inputs (features)
- Input: $\mathbf{X} \in \mathbb{R}^{n \times d}$
- One hidden layer with h hidden units.
- Hidden variable: $\mathbf{H} \in \mathbb{R}^{n \times h}$
- Hidden layer
  - weights: $\mathbf{W}^{(1)} \in \mathbb{R}^{d \times h}$
  - biases: $\mathbf{b}^{(1)} \in \mathbb{R}^{1 \times h}$
- Output layer with q units.
- Output layer
  - weights: $\mathbf{W}^{(2)} \in \mathbb{R}^{h \times q}$
  - biases: $\mathbf{b}^{(2)} \in \mathbb{R}^{1 \times q}$



Single Hidden Layer

- Outputs $\mathbf{O} \in \mathbb{R}^{n \times q}$

$$\mathbf{H} = \mathbf{X}\mathbf{W}^{(1)} + \mathbf{b}^{(1)},$$
$$\mathbf{O} = \mathbf{H}\mathbf{W}^{(2)} + \mathbf{b}^{(2)}.$$



- Adding hidden layer requires tracking and updating additional sets of parameters
- Rewriting :

$$\mathbf{O} = (\mathbf{X}\mathbf{W}^{(1)} + \mathbf{b}^{(1)})\mathbf{W}^{(2)} + \mathbf{b}^{(2)} = \mathbf{X}\mathbf{W}^{(1)}\mathbf{W}^{(2)} + \mathbf{b}^{(1)}\mathbf{W}^{(2)} + \mathbf{b}^{(2)} = \mathbf{X}\mathbf{W} + \mathbf{b}$$

- Can be represented by a single layer neural network

Single Hidden Layer

- Outputs $\mathbf{O} \in \mathbb{R}^{n \times q}$

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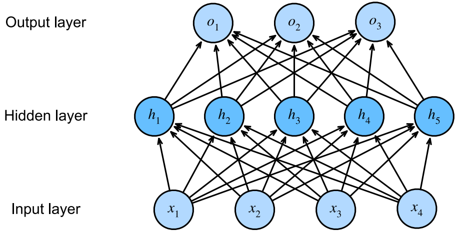

- Adding hidden layer requires tracking and updating additional sets of parameters
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- Can be represented by a single layer neural network
- Still a Linear Model

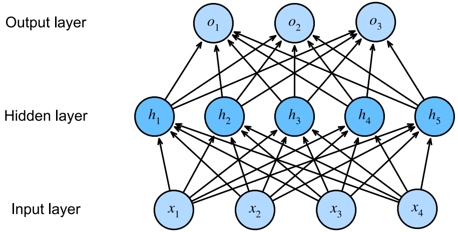
Non Linear Activation Function

- A nonlinear activation function σ should be applied to each hidden unit following the affine transformation
- With activation functions, it is no longer possible to collapse our MLP into a linear model

$$\mathbf{H} = \sigma(\mathbf{X}\mathbf{W}^{(1)} + \mathbf{b}^{(1)}),$$
$$\mathbf{O} = \mathbf{H}\mathbf{W}^{(2)} + \mathbf{b}^{(2)}.$$

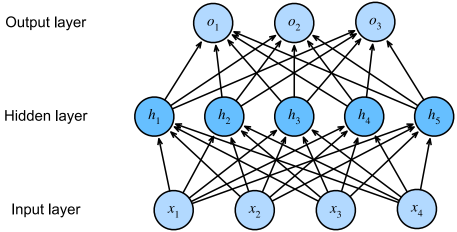
- In general, with more hidden layers

$$\mathbf{H}^{(1)} = \sigma_1(\mathbf{X}\mathbf{W}^{(1)} + \mathbf{b}^{(1)})$$
$$\mathbf{H}^{(2)} = \sigma_2(\mathbf{H}^{(1)}\mathbf{W}^{(2)} + \mathbf{b}^{(2)})$$



Activation Functions

- Decide whether a neuron should be activated or not by calculating the weighted sum and further adding bias with it.
- Need to be differentiable operators to transform input signals to outputs
- Adds non-linearity to the model



ReLU Function

- Rectified Linear Unit (ReLU): simple nonlinear transformation
- Given an element x, the function is defined as the maximum of that element and 0
- $\text{ReLU}(x) = \max(x, 0)$
- ReLU is piecewise linear
- Deivative: $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$, undefined at 0.
- At $x = 0$, we default to the left-hand-side derivative

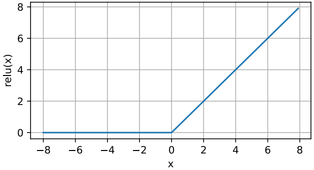


Figure: ReLU

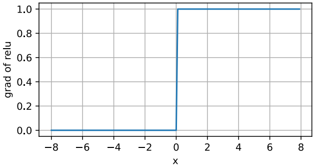
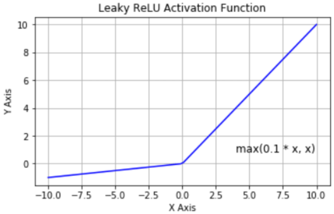


Figure: Derivative of ReLU

Generalization of ReLU

- Generalizations are based on the principle that models are easier to optimize if their behavior is closer to linear
- Add non-linear slope: α_i, when $x_i < 0$
- $h_i = g(\boldsymbol{x}, \boldsymbol{\alpha})_i = \max(0, x_i) + \alpha_i \min(0, x_i)$
- *Leaky ReLU*: Fixes α_i to a small value like 0.01 (Maas et al., 2013)
- *parameterized ReLU (pReLU)*: Treats α_i as a learnable parameter (He et al., 2015)
- *Maxout units*: Divides $\boldsymbol{x}$ into groups of k values (Goodfellow et al., 2013) $g(\boldsymbol{x})_i = \max_{j \in \mathbb{G}^{(i)}} x_j$



Sigmoid Function

- Transforms its inputs from domain $\mathbb{R}$ to outputs that lie on the interval (0, 1).

$$\text{sigmoid}(x) = \frac{1}{1+\exp(-x)}$$

- Smooth, differentiable approximation to a thresholding unit
- Derivative of the Sigmoid function:

$$\frac{d}{dx}\text{sigmoid}(x) = \frac{\exp(-x)}{(1+\exp(-x))^2} = \text{sigmoid}(x)\left(1-\text{sigmoid}(x)\right).$$

- When the input is 0, the derivative of the sigmoid function reaches a maximum of 0.25.

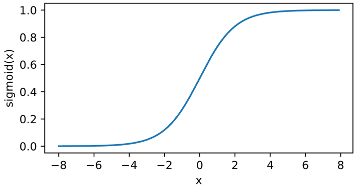


Figure: Sigmoid

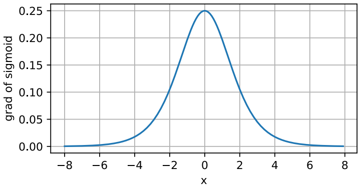


Figure: Gradient of Sigmoid

Tanh (Hyperbolic Tangent) Function

- Transforms its inputs from domain $\mathbb{R}$ to outputs that lie on the interval (-1, 1).

$$\tanh(x) = \frac{1-\exp(-2x)}{1+\exp(-2x)}$$

- As the input nears 0, it approaches a linear transformation
- Derivative of the Tanh function:

$$\frac{d}{dx}\tanh(x) = 1 - \tanh^2(x)$$

- As the input nears 0, the derivative of the tanh function approaches a maximum of 1

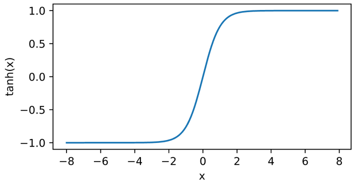


Figure: Tanh

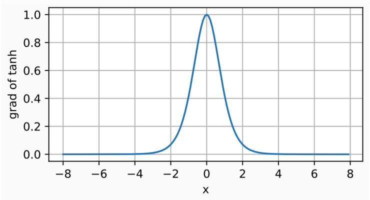
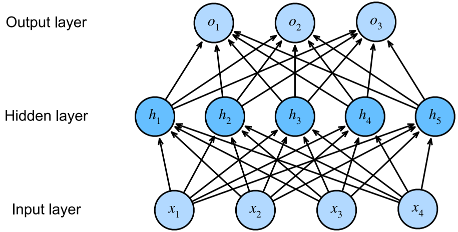


Figure: Gradient of Tanh

Multiclass Classification

Outputs: $y_1, y_2, \ldots, y_k = \text{softmax}(o_1, o_2, \ldots, o_k)$

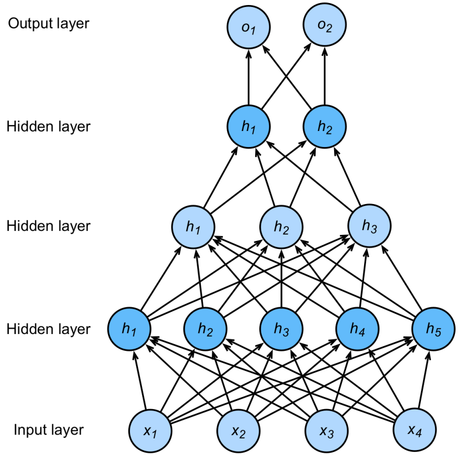
$$\mathbf{h} = \sigma(\mathbf{W}_1 \mathbf{x} + \mathbf{b}_1)$$
$$\mathbf{o} = \mathbf{w}_2^T \mathbf{h} + \mathbf{b}_2$$
$$\mathbf{y} = \text{softmax}(o)$$



Multiclass Classification

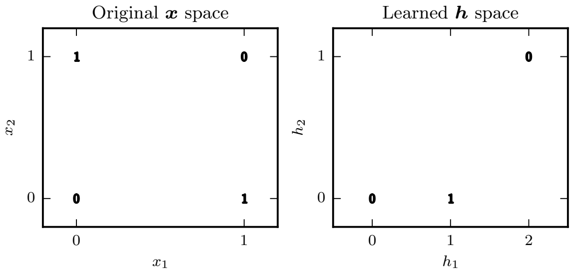
$$y_1, y_2, \ldots, y_k = \text{softmax}(o_1, o_2, \ldots, o_k)$$

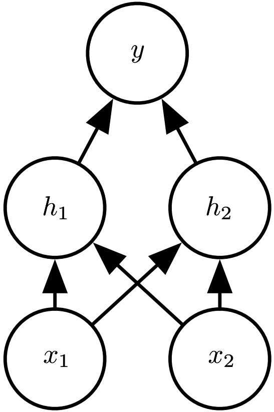
$$\mathbf{h}_1 = \sigma(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1)$$
$$\mathbf{h}_2 = \sigma(\mathbf{W}_2\mathbf{h}_1 + \mathbf{b}_2)$$
$$\mathbf{h}_3 = \sigma(\mathbf{W}_3\mathbf{h}_2 + \mathbf{b}_3)$$
$$\mathbf{o} = \mathbf{W}_4\mathbf{h}_3 + \mathbf{b}_4$$



Solving the XOR Problem

- One hidden layer with two hidden units
- MSE loss function
  $J(\boldsymbol{\theta}) = \frac{1}{4} \sum_{\boldsymbol{x} \in \mathbb{X}} (f^*(\boldsymbol{x}) - f(\boldsymbol{x}; \boldsymbol{\theta}))^2$
- ReLU Activation function





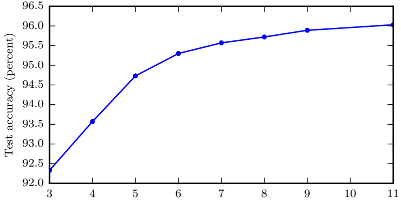
Architecture Basics

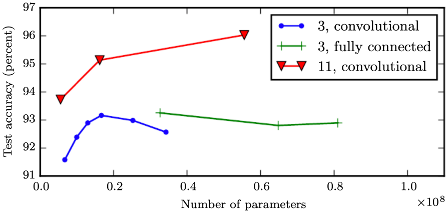
The Universal Approximation Theorem - [Hornik et al., 1989; Cybenko, 1989]

One hidden layer is enough to represent (not learn) an approximation of any function to an arbitrary degree of accuracy

So why deeper Neural Network?

- Shallow net may need (exponentially) more width
- Shallow net may overfit more





Generalization Error

- *Generalization*: ability to perform well on previously unobserved inputs
- *Training Error*: prediction error based on data available at training (like optimization)
- *Generalization Error / Testing Error*: error on data not seen during testing
- Machine Learning vs Optimization: In ML, we aim to minimize the generalization loss
- Linear Regression Training Error:

$$\frac{1}{m^{(\text{train})}} \left\| \boldsymbol{X}^{(\text{train})} \boldsymbol{w} - \boldsymbol{y}^{(\text{train})} \right\|_2^2$$

- Linear Regression Testing Error:

$$\frac{1}{m^{(\text{test})}} \left\| \boldsymbol{X}^{(\text{test})} \boldsymbol{w} - \boldsymbol{y}^{(\text{test})} \right\|_2^2$$

- Machine Learning vs Optimization: In ML, we aim to minimize the generalization loss

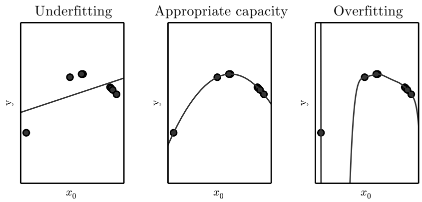
Data Distribution

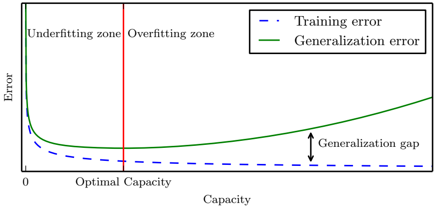
How can we affect performance on the test set when we get to observe only the training set?

- Assumptions:
  - training and test sets are not collected arbitrarily
  - Independent and identically distributed (i.i.d. or IID)
    - + examples in each dataset are independent from each other
    - + train and test set are identically distributed (drawn from same probability distribution)
- Under these assumptions: expected training set error = expected test set error
- Training process: training dataset is used to choose the parameters that minimize the chosen loss function
- Test error $\geq$ Train error
- Goal of ML algorithm:
  - Make the training error small
  - Make the gap between training and test error small

Underfitting vs Overfitting

- *Underfitting*: the model is not able to obtain a sufficiently low error value on the training set
- *Overfitting*: the gap between the training error and test error is too large
- *Capacity*: ability to fit a wide variety of functions





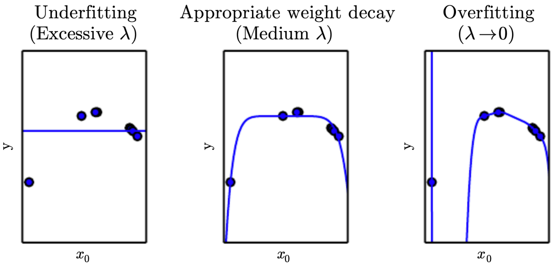
- *No Free Lunch Theorem:* [Wolpert, 1996] averaged over *all possible* data generating distributions, every classification algorithm has the same error rate when classifying previously unobserved points

Regularization

- *Weight Decay*: Reduce model complexity by limiting value range
- Original loss of linear regression:
  - minimize the prediction loss on the training labels
  - $L(\mathbf{w}, b) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - y^{(i)} \right)^2 .$
- Regularized loss:
  - minimizing the sum of the prediction loss and the penalty term
  - $L(\mathbf{w}, b) + \frac{\lambda}{2} \|\mathbf{w}\|^2$
- Weight Updates:
  - $\mathbf{w} \leftarrow (1 - \eta\lambda) \mathbf{w} - \frac{\eta}{|\mathcal{B}|} \sum_{i \in \mathcal{B}} \mathbf{x}^{(i)} \left(\mathbf{w}^\top \mathbf{x}^{(i)} + b - y^{(i)} \right) .$
- Optimization algorithm *decays the weight* at each step of training

Effect of Regularization

- Train a high-degree polynomial regression model

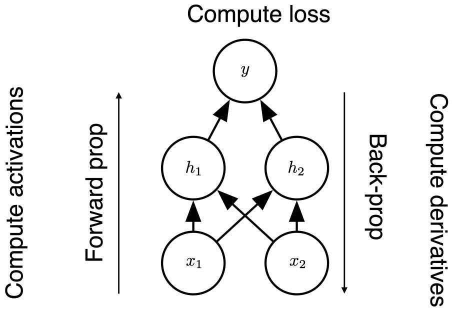


Regularization is any modification we make to a learning algorithm that is intended to reduce its generalization error but not its training error.

Training Neural Networks

Minibatch Stochastic Gradient Descent

- Forward Propagation: calculation and storage of intermediate variables (including outputs) in order from the input layer to the output layer
- Backward Propagation: method of calculating the gradient of neural network parameters for the weights to be updated

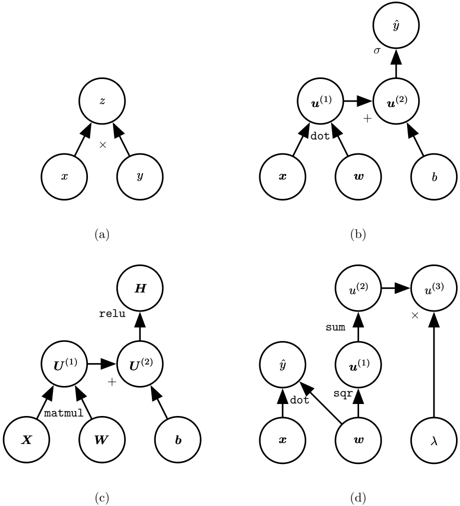


Forward Propagation

- Simple Assumptions:
  - Input: $\mathbf{x} \in \mathbb{R}^d$
  - Intermediate variable: $\mathbf{z} \in \mathbb{R}^h$
  - Weight parameter of the hidden layer: $\mathbf{W}^{(1)} \in \mathbb{R}^{h \times d}$
  - Before activation: $\mathbf{z} = \mathbf{W}^{(1)}\mathbf{x}$, when hidden terms do not have a bias
  - After passing through activation function (ϕ):
    hidden variable $\mathbf{h} = \phi(\mathbf{z})$
  - Weight parameter of output layer: $\mathbf{W}^{(2)} \in \mathbb{R}^{q \times h}$
  - Output layer variable: $\mathbf{o} = \mathbf{W}^{(2)}\mathbf{h}$
  - Loss function: $L = l(\mathbf{o}, y)$

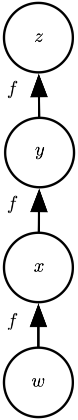
Computational Graph

- A formal way to represent math in graph theory
- Each node represents a variable (a scalar, vector, matrix, tensor) or an operation
- Operation: a simple function of one or more variables
- Directed edge: input and output relations of operators and variables
- Example
  - (a) Multiplication
  - (b) Logistic regression
  - (c) ReLU layer
  - (d) Linear regression and weight decay



Backpropagation

- A method of calculating the gradient of neural network parameters
- Computes the chain rule of calculus
- Stores any intermediate variables (partial derivatives)
- Done by table filling
- Brings down the complexity from $O\left(n^2\right)$ to $O\left(n\right)$



$$\begin{aligned}&\frac{\partial z}{\partial w}\\ &=\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial w}\\ &=f'(y)f'(x)f'(w)\\ &=f'(f(f(w)))f'(f(w))f'(w)\end{aligned}$$

Back-prop avoids computing this twice

Backpropagation

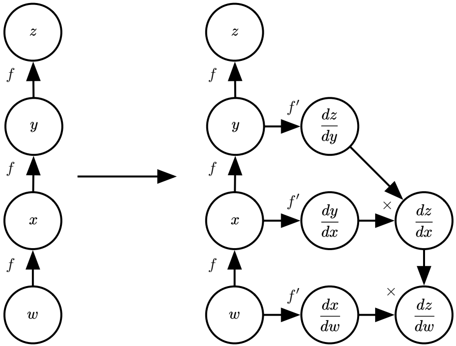


Figure: Symbol-to-symbol approach to computing derivatives

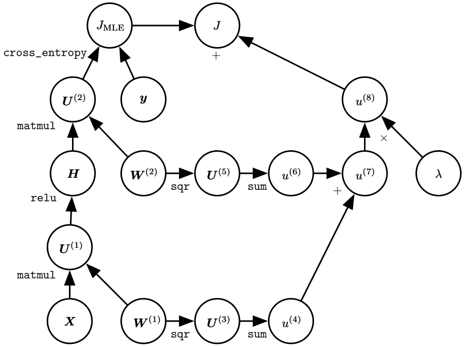
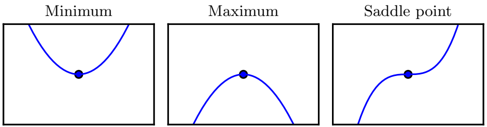
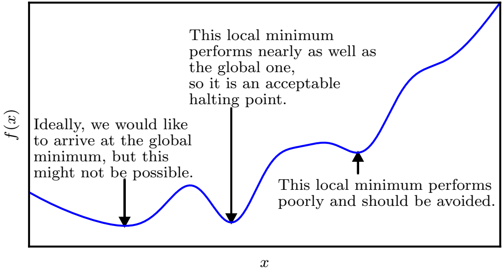


Figure: Cross entropy loss for forward propagation

Loss Function and Gradient Descent

Loss function should be convex.

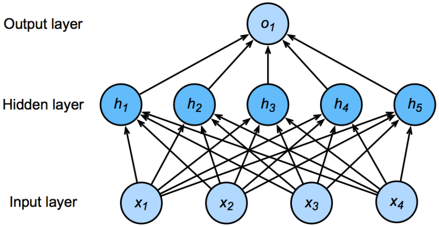




Convolutional Neural Network

Scalability of Feed Forward Network

- Example Application: Distinguish cat and dog
  - Phone camera (12MP)
  - RGB image has 36M elements
  - Single hidden layer of 100 hidden neurons
  - The model size is 3.6 Billion parameters
  - Infeasible to learn these many parameters



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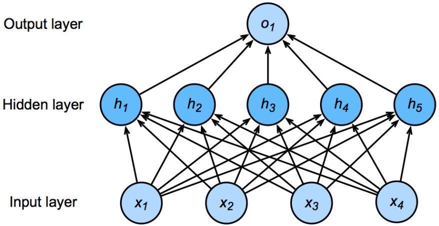
- Reduce size of image (1MP)
  - Hidden layer size is underestimated
  - Requires enormous dataset



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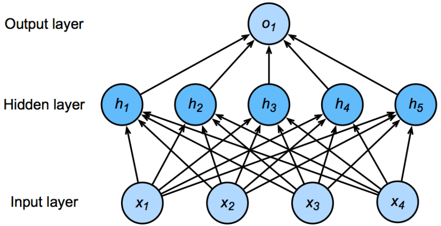
- Reduce size of image (1MP)
  - Hidden layer size is underestimated
  - Requires enormous dataset
- Convolutional Neural Networks(CNNs) are designed to build efficient models



Two Principles

Translation Invariance: network should respond similarly to the same patch, regardless of where it appears in the image

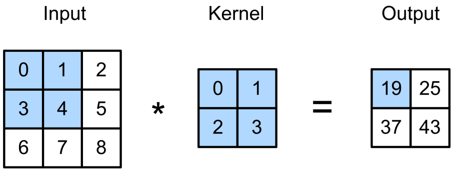
Locality: network should focus on local regions, without regard for the contents of the image in distant regions

Convolutional Neural Network

- Neural networks that use *convolution* in place of general matrix multiplication in at least one of their layers
- Convolution:
  - $s(t) = \int x(a)w(t-a)da$
  - $s(t) = (x * w)(t)$ – denoted by asterisk
  - In ML terminology, x is input and w is kernel or filter
  - Measure of the overlap between f and g when one function is "flipped" and shifted by t
  - Discrete representation:
    $s(t) = (x * w)(t) = \sum_{a=-\infty}^{\infty} x(a)w(t-a)$
  - On finite input:
    $s(t) = (x * w)(t) = \sum_a x(a)w(t-a)$
  - On two dimensional input I and kernel K,
    $S(i,j) = (I * K)(i,j) = \sum_m \sum_n I(m,n)K(i-m,j-n)$
    $S(i,j) = (K * I)(i,j) = \sum_m \sum_n I(i-m,j-n)K(m,n)$ – commutative, easy to implement

Convolution vs Cross Correlation

- Cross Correlation:
  $S(i,j) = (I * K)(i,j) = \sum_m \sum_n I(i+m, j+n)K(m,n)$
- Same as convolution, without flipping kernel
- No difference in practice due to symmetry



$$0 \times 0 + 1 \times 1 + 3 \times 2 + 4 \times 3 = 19,$$
$$1 \times 0 + 2 \times 1 + 4 \times 2 + 5 \times 3 = 25,$$
$$3 \times 0 + 4 \times 1 + 6 \times 2 + 7 \times 3 = 37,$$
$$4 \times 0 + 5 \times 1 + 7 \times 2 + 8 \times 3 = 43.$$

Figure: Cross Correlation.

Convolution Layer

- Input Matrix: $\mathbf{I} : n_h \times n_w$
- Kernel Matrix: $\mathbf{K} : k_h \times k_w$
- Scalar Bias: b
- Output Matrix: $\mathbf{S} : (n_h - k_h + 1) \times (n_w - k_w + 1)$
- $\mathbf{S} = \mathbf{I} \star \mathbf{K} + b$
- $\mathbf{K}$ and b are learnable parameters
- Common choice: k_h or k_w are odd (1, 3, 5, 7).

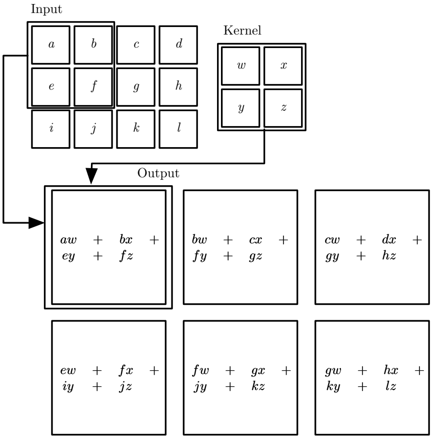


Figure: 2-D convolution without kernel-flipping.

Padding

- Given a 32 x 32 input image
- Apply convolutional layer with 5 x 5 kernel
- 28 x 28 output with 1 layer
- 4 x 4 output with 7 layers
- Shape decreases faster with larger kernels
- Shape reduces from $n_h \times n_w$ to $(n_h - k_h + 1) \times (n_w - k_w + 1)$

Padding

- Given a 32 x 32 input image
- Apply convolutional layer with 5 x 5 kernel
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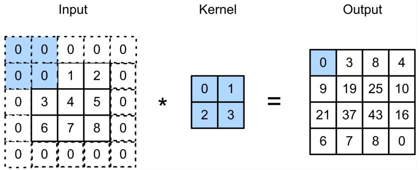
- Shape decreases faster with larger kernels
- Shape reduces from $n_h \times n_w$ to $(n_h - k_h + 1) \times (n_w - k_w + 1)$

- *Padding* adds rows/columns around input



- Padding p_h rows and p_w columns, output shape will be $(n_h - k_h + p_h + 1) \times (n_w - k_w + p_w + 1)$
- Common choice: $p_h = k_h - 1$, $p_w = k_w - 1$
  - Odd k_h: pad $p_h/2$ on both sides
  - Even k_h: pad $\lceil p_h/2 \rceil$ on top, $\lfloor p_h/2 \rfloor$ on bottom

Stride

- Padding reduces shape linearly with number of layers:
  - a 224×224 input with a 5×5 kernel, needs 44 layers to reduce the shape to 4×4
  - Still large amount of computation
- *Stride:* number of rows and columns traversed per slide

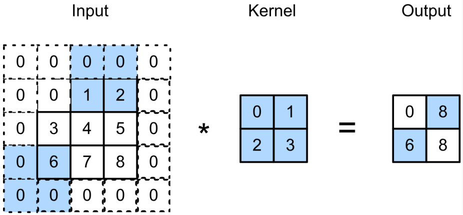


Figure: Stride 3 and 2 for height and width

- With strides s_h and s_w, output shape:
  $\lfloor (n_h - k_h + p_h + s_h)/s_h \rfloor \times \lfloor (n_w - k_w + p_w + s_w)/s_w \rfloor$
- With $p_h = k_h - 1, p_w = k_w - 1$
  $\lfloor (n_h + s_h - 1)/s_h \rfloor \times \lfloor (n_w + s_w - 1)/s_w \rfloor$
- If input height/width are divisible by strides
  $(n_h/s_h) \times (n_w/s_w)$

Convolution with stride

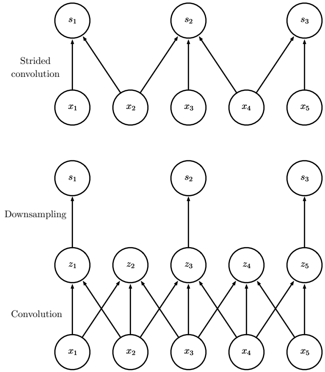


Figure: Equivalent Models

Multiple Input Channels

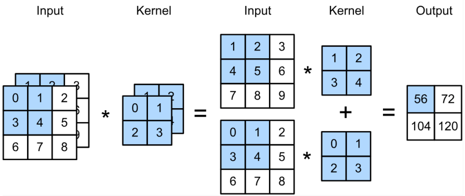
- *Channel*: another dimension in tensor
  - Wireless Signals: Real/Imaginary
  - Image: RGB
- Have a kernel for each channel, and then sum results over channels

  Input: $\mathbf{X} : c_i \times n_h \times n_w$

  Kernel: $\mathbf{W} : c_i \times k_h \times k_w$

  Output: $\mathbf{Y} : m_h \times m_w$

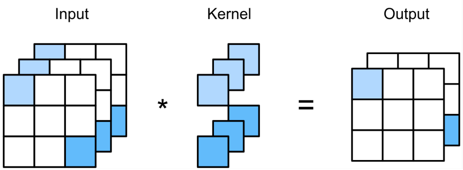
  $\mathbf{Y} = \sum_{i=0}^{c_i} \mathbf{X}_{i,:,:} \star \mathbf{W}_{i,:,:}$



$$(1\times1+2\times2+4\times3+5\times4)+(0\times0+1\times1+3\times2+4\times3) = 56$$

Multiple Output Channels

- Each output channel may recognize a different pattern
- A kernel for every output channel
  Input: $\mathbf{X} : c_i \times n_h \times n_w$
  Kernel: $\mathbf{W} : c_o \times c_i \times k_h \times k_w$
  Output: $\mathbf{Y} : c_o \times m_h \times m_w$
  $\mathbf{Y}_{i,:,:} = \mathbf{X} \star \mathbf{W}_{i,:,:,:}$ for for $i = 1, \ldots, c_o$



Pooling

Convolution is sensitive to position

```
X    [[1. 1. 0. 0. 0.        Y    [[ 0.  1.  0.  0.
      [1. 1. 0. 0. 0.              [ 0.  1.  0.  0.
      [1. 1. 0. 0. 0.              [ 0.  1.  0.  0.
      [1. 1. 0. 0. 0.              [ 0.  1.  0.  0.
```

Figure: Example: Vertical Edge Detection

- *Pooling* over spatial regions produces invariance to translation
- Summarizes the responses over a whole neighborhood
- Calculate either maximum or average value of the elements in the pooling window

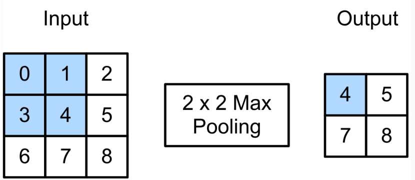
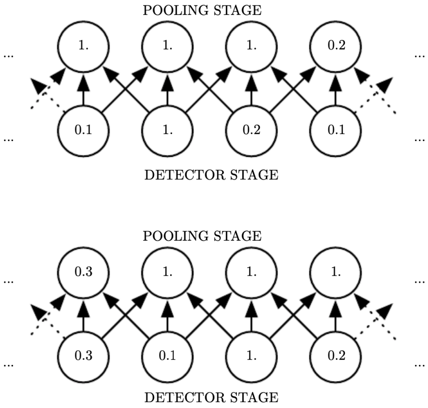
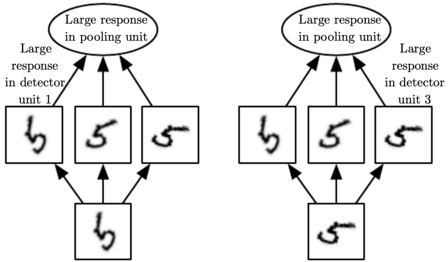


Figure: Example: Max Pool

Pooling introduces invariance





Pooling with downsampling

- Have similar padding and stride as convolutional layers
- Stride introduces downsampling
- No learnable parameters
- Apply pooling for each input channel to obtain the corresponding output channel
  No. output channels = No. input channels

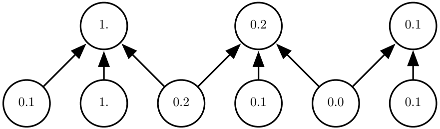
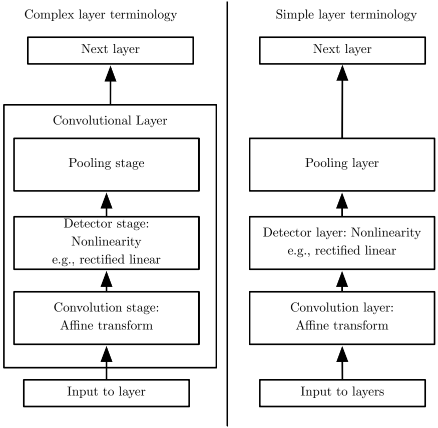


Figure: Max-pooling with a pool window of 3 and a stride of 2

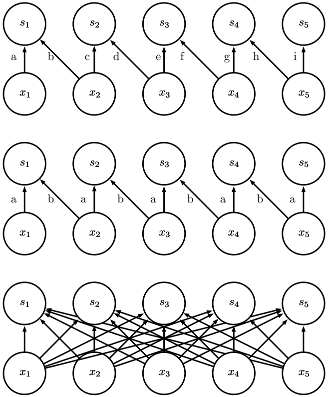
Convolutional Layer

- Consists of three stages:
  - *Convolution*: several convolutions in parallel to produce a set of linear activations
  - *Detector*: each linear activation is run through a nonlinear activation function
  - *Pooling*: replaces the output with a summary statistic of the nearby outputs

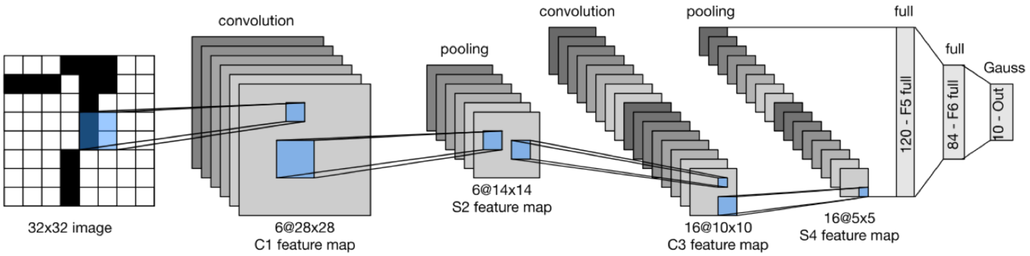


Convolution vs Fully Connected Network

- Local Connections, no sharing of parameters
- Convolution
- Fully Connected



LeNet [Yann LeCun et al. in 1989]



Modern Convolutional Neural Networks

- LeNet (the first convolutional neural network)
- AlexNet [2012]
  - More of everything
  - ReLu, Dropout, Invariances
- VGG [2014]
  - Repeated blocks: even more of everything (narrower and deeper)
- NiN [2013]
  - 1x1 convolutions + global pooling instead of dense
- GoogLeNet [2015]
  - Inception block: explores parallel paths with different kernels
- Residual network (ResNet) [2016]
  - Residual block: computes residual mapping $f(\mathbf{x}) - \mathbf{x}$ in forward path

Sequence Modeling

Data

- So far . . .
- Collect observation pairs $(x_i, y_i) \sim p(x, y)$ for training
- Estimate for $y|x \sim p(y|x)$ unseen $x' \sim p(x)$
- Examples:
  - Images & objects
  - Regression problem
  - House & house prices
- *The order of the data did not matter*

Dependence on time

- Natural Language Processing
- Stock price prediction
- Movie prediction
- Wireless Applications:
  - Change in wireless channel
  - Channel Coding
  - Secure waveform generation
  - User mobility/beam prediction

Data usually is not independently and identically distributed (IID)

Autoregressive Model

- Observations from previous time steps are input to a regression model
- Linear Regression: $Y = wX + b$
- Number of inputs $x_{t-1}, \ldots, x_1$ vary with t
- First Strategy:
  - Long sequence is not required
  - $x_{t-1}, \ldots, x_{t-\tau}$ are used
- Second Strategy:
  - Keep summary h_t and update it every step
  - Latent Autoregressive model, where $\hat{x}_t = P(x_t \mid h_t)$ and $h_t = g(h_{t-1}, x_{t-1})$

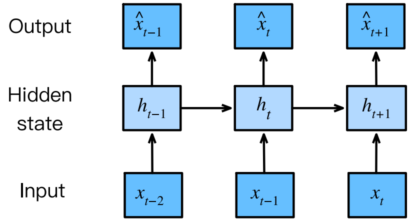


Figure: Latent Autoregressive Model

Sequence Model

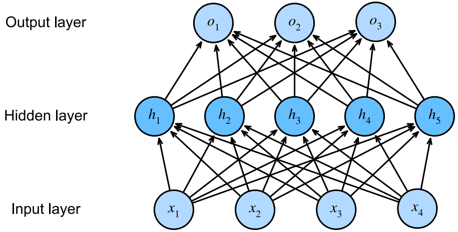
- Input x_t, where t is discrete step in time $t \in \mathbb{Z}^+$
- Dependent random variables
  $(x_1, \dots x_T) \sim p(x)$
- Conditional Probability Expansion
  $p(x) = p(x_1) \cdot p(x_2|x_1) \cdot p(x_3|x_1, x_2) \cdot \dots p(x_T|x_1, \dots x_{T-1})$
- Could also find reverse direction
  $p(x) = p(x_T) \cdot p(x_{T-1}|x_T) \cdot p(x_{T-2}|x_{T-1}, x_T) \cdot \dots p(x_1|x_2, \dots x_T)$
- Causality (physics) prevents the reverse direction (future events cannot influence the past)
- Train with sequence of data (not randomized)

Recurrent Neural Networks

- Neural Networks with hidden states
- Hidden state (h_{t-1}) capable of storing the sequence information $P(x_t \mid x_{t-1}, \ldots, x_1) \approx P(x_t \mid h_{t-1})$
- Hidden state computed as: $h_t = f(x_t, h_{t-1})$.
- Function f approximates all hidden information
- *Hidden State* is different from *Hidden Layer*

Neural Network without Hidden State

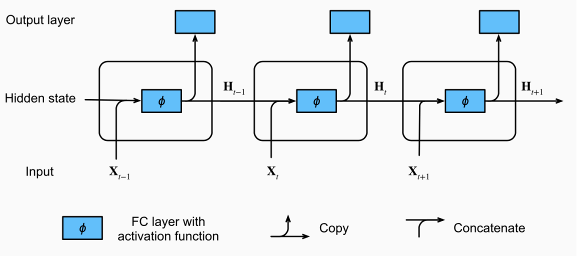
- With batch size n and width d input $\mathbf{X} \in \mathbb{R}^{n \times d}$
- Hidden layer output: $\mathbf{H} \in \mathbb{R}^{n \times h}$
  $\mathbf{H} = \phi(\mathbf{X}\mathbf{W}_{xh} + \mathbf{b}_h)$
- Activation function ϕ
- Weight and bias of h hidden units:
  $\mathbf{W}_{xh} \in \mathbb{R}^{d \times h}$, $\mathbf{b}_h \in \mathbb{R}^{1 \times h}$
- Output: $\mathbf{O} \in \mathbb{R}^{n \times q}$
  $\mathbf{O} = \mathbf{H}\mathbf{W}_{hq} + \mathbf{b}_q$,
- Weight and bias of q output units:
  $\mathbf{W}_{hq} \in \mathbb{R}^{h \times q}$, $\mathbf{b}_q \in \mathbb{R}^{1 \times q}$



- This is similar to the autoregression problem

Recurrent Neural Network - with hidden state

- Input $\mathbf{X}_t \in \mathbb{R}^{n \times d}$
- For a minibatch of size n, each row of $\mathbf{X}_t$ corresponds to one example at time step t from the sequence
- Hidden variable of current time step depends on input of the current time step and hidden variable of the previous time step
  $$\mathbf{H}_t = \phi(\mathbf{X}_t \mathbf{W}_{xh} + \underbrace{\mathbf{H}_{t-1} \mathbf{W}_{hh}}_{\text{hidden state dependency}} + \mathbf{b}_h)$$
- Output:
  $$\mathbf{O}_t = \mathbf{H}_t \mathbf{W}_{hq} + \mathbf{b}_q.$$



RNN for Natural Language Processing

- In 2003, Bengio et al. first proposed to use a neural network for language modeling
- Tokenize text into characters

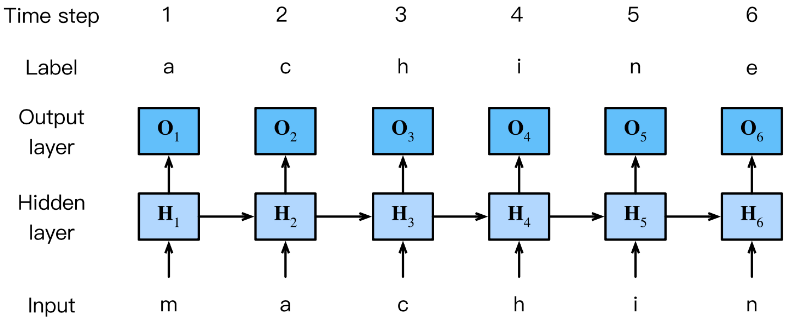


Figure: Character level language model

Loss Function for Softmax - Revisited

- The softmax function outputs a vector $\hat{\mathbf{y}}$, which can be interpreted as estimated conditional probabilities of each class given any input $\mathbf{x}$

$$\hat{y} = P(y = \text{Category } 1 \mid \mathbf{x})$$

- For the entire dataset, $P(\mathbf{Y} \mid \mathbf{X}) = \prod_{i=1}^{n} P(\mathbf{y}^{(i)} \mid \mathbf{x}^{(i)})$
- According to maximum likelihood estimation, we maximize $P(\mathbf{Y} \mid \mathbf{X})$
- This is equivalent to minimizing the negative log-likelihood

$$-\log P(\mathbf{Y} \mid \mathbf{X}) = \sum_{i=1}^{n} -\log P(\mathbf{y}^{(i)} \mid \mathbf{x}^{(i)}) = \sum_{i=1}^{n} l(\mathbf{y}^{(i)}, \hat{\mathbf{y}}^{(i)})$$

- For any pair of label $\mathbf{y}$ and model prediction $\hat{\mathbf{y}}$ over q classes, the loss function l is

$$l(\mathbf{y}, \hat{\mathbf{y}}) = -\sum_{j=1}^{q} y_j \log \hat{y}_j \qquad \text{– } \textit{Cross Entropy Loss}$$

- $\mathbf{y}$ one-hot vector of length q, so sum over all its coordinates j vanishes for all except one term
- $\hat{y}_j$ are probabilities, so their logarithms are never greater than 0.

Loss Function for RNN

- Which token to choose in the next time step?
- Quality of the model can be measured by computing the likelihood of the sequence
- Issue: shorter sequences are much more likely to occur than the longer ones
- Solution: Cross-entropy loss *averaged* over all the n tokens of a sequence
  $\frac{1}{n}\sum_{t=1}^{n} -\log P(x_t \mid x_{t-1}, \ldots, x_1)$
- *Perplexity* in NLP: Harmonic mean of the number of real choices
  $\exp\left(-\frac{1}{n}\sum_{t=1}^{n} \log P(x_t \mid x_{t-1}, \ldots, x_1)\right)$
  - Best case: perplexity is 1.
  - Worst case: perplexity is 0.
  - Baseline: predicts a uniform distribution over all the available tokens

Gradients in RNN

- Hidden and Output Layer: $h_t = f(x_t, h_{t-1}, w_h), \quad o_t = g(h_t, w_o)$
- Chain of values that depend on recurrent computation: $\{\dots, (x_{t-1}, h_{t-1}, o_{t-1}), (x_t, h_t, o_t), \dots\}$
- Forward propagation: Compute (x_t, h_t, o_t) at each time step
- Difference between output o_t and label y_t is:
  $L(x_1, \dots, x_T, y_1, \dots, y_T, w_h, w_o) = \frac{1}{T}\sum_{t=1}^{T} l(y_t, o_t)$
- Backward propagation (by Chain Rule):

$$\begin{aligned}\frac{\partial L}{\partial w_h} &= \frac{1}{T}\sum_{t=1}^{T}\frac{\partial l(y_t, o_t)}{\partial w_h} \\ &= \frac{1}{T}\sum_{t=1}^{T}\frac{\partial l(y_t, o_t)}{\partial o_t}\frac{\partial g(h_t, w_o)}{\partial h_t}\underbrace{\frac{\partial h_t}{\partial w_h}}_{\text{Recurrent computation needed}}\end{aligned}$$

- h_t depends on h_{t-1} and w_h

Backpropagation Through Time

- Note: $h_t = f(x_t, h_{t-1}, w_h)$
- Third Term of $\frac{\partial L}{\partial w_h}$:

$$\frac{\partial h_t}{\partial w_h} = \frac{\partial f(x_t, h_{t-1}, w_h)}{\partial w_h} + \frac{\partial f(x_t, h_{t-1}, w_h)}{\partial h_{t-1}} \frac{\partial h_{t-1}}{\partial w_h}$$

- Can be written as three sequences:

$$a_t = b_t + \sum_{i=1}^{t-1} \left(\prod_{j=i+1}^{t} c_j \right) b_i$$

$$a_t = \frac{\partial h_t}{\partial w_h},$$

$$b_t = \frac{\partial f(x_t, h_{t-1}, w_h)}{\partial w_h},$$

$$c_t = \frac{\partial f(x_t, h_{t-1}, w_h)}{\partial h_{t-1}}$$

Backpropagation Through Time

- BPTT:

$$\frac{\partial h_t}{\partial w_h} = \frac{\partial f(x_t, h_{t-1}, w_h)}{\partial w_h} + \sum_{i=1}^{t-1} \left(\prod_{j=i+1}^{t} \frac{\partial f(x_j, h_{j-1}, w_h)}{\partial h_{j-1}} \right) \frac{\partial f(x_i, h_{i-1}, w_h)}{\partial w_h}.$$

- Chain rule can be used to compute $\partial h_t / \partial w_h$ recursively
- Chain gets long with t
- Solution: truncate time steps [2002]

Backpropagation Through Time

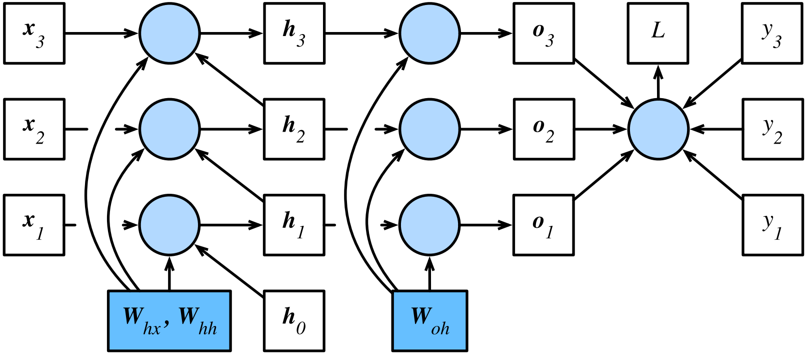
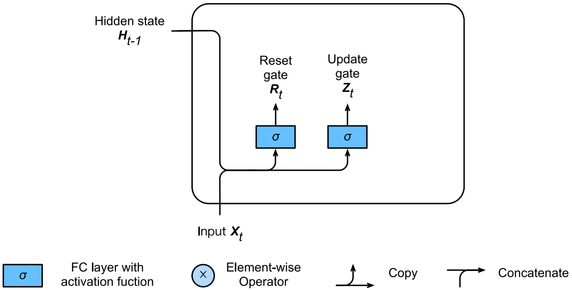


Figure: Computational graph showing dependencies for an RNN model with three time steps

Gated Recurrent Unit (GRU)

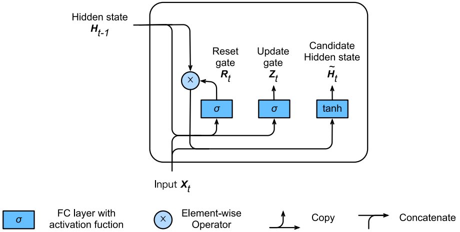
- Not all observations are equally relevant
- Engineered Gates: Reset and Update
  - Reset: mechanism to forget
  - Update: mechanism to pay attention

$$R_t = \sigma(X_t W_{xr} + H_{t-1} W_{hr} + b_r),$$
$$Z_t = \sigma(X_t W_{xz} + H_{t-1} W_{hz} + b_z)$$

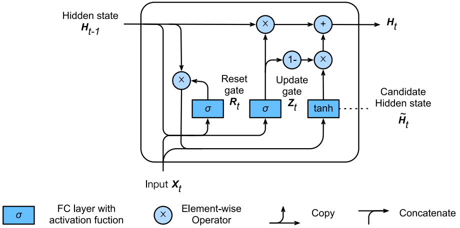


Gated Recurrent Unit (GRU)

- Candidate Hidden State
  $\tilde{\mathbf{H}}_t = \tanh(\mathbf{X}_t \mathbf{W}_{xh} + (\mathbf{R}_t \odot \mathbf{H}_{t-1})\,\mathbf{W}_{hh} + \mathbf{b}_h)$

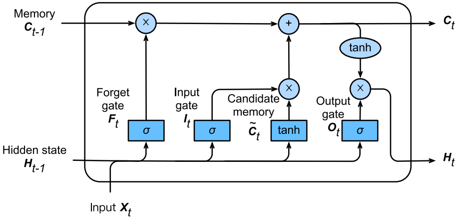


- Hidden State (incorporates update)
  $\mathbf{H}_t = \mathbf{Z}_t \odot \mathbf{H}_{t-1} + (1 - \mathbf{Z}_t) \odot \tilde{\mathbf{H}}_t$



Long Short Term Memory (LSTM)

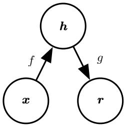
$$I_t = \sigma(X_t W_{xi} + H_{t-1} W_{hi} + b_i)$$
$$F_t = \sigma(X_t W_{xf} + H_{t-1} W_{hf} + b_f)$$
$$O_t = \sigma(X_t W_{xo} + H_{t-1} W_{ho} + b_o)$$
$$\tilde{C}_t = \tanh(X_t W_{xc} + H_{t-1} W_{hc} + b_c)$$
$$C_t = F_t \odot C_{t-1} + I_t \odot \tilde{C}_t$$
$$H_t = O_t \odot \tanh(C_t)$$

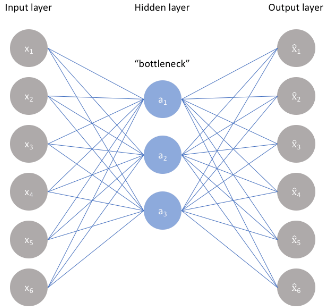


Other Networks

Autoencoder

- Autoencoder:
  - an encoder function f: converts the input data into a different representation
  - a decoder function g: converts the new representation back into the original format
- Bottleneck: Compressed knowledge representation
- Unsupervised learning
- Reconstruction Error: $L(\boldsymbol{x}, \boldsymbol{r}) = L(\boldsymbol{x}, g(f(\boldsymbol{x})))$
- Dimensionality Reduction
- Without non-linear activation functions, performs Principal Component Analysis (PCA)
- Denoising Autoencoder: Input $\hat{\boldsymbol{x}}$ is corrupted $\boldsymbol{x}$, and reconstruction loss is $L(\boldsymbol{x}, g(f(\hat{\boldsymbol{x}})))$



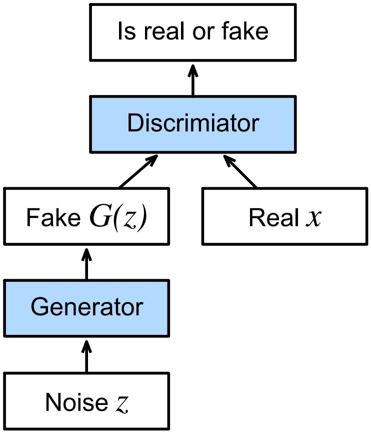


Generative Adversarial Network

- Discriminative Models: Given input $\boldsymbol{X}$, predict label $\boldsymbol{Y}$
  - Estimates $P(\boldsymbol{Y}|\boldsymbol{X})$
- Discriminative models have limitations:
  - Can't model $P(\boldsymbol{X})$
  - Can't sample from $P(\boldsymbol{X})$, i.e. can't generate new data
- Generative Model:
  - Can model $P(\boldsymbol{X})$
  - Can generate $P(\boldsymbol{X})$

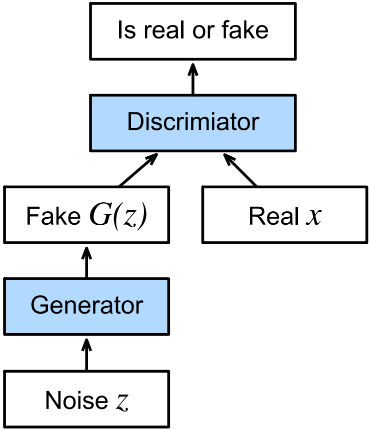
Generative Adversarial Network [Goodfellow et al., 2014]

- Leverages power of discriminative models to get good generative models
- A generator is good if we cannot tell fake data apart from real data
- *Statistical Tests*: identifies whether the two datasets come from the same distribution
- *GAN*: Based on the feedback from discriminator, creates a generator until it generates something that resembles the real data
- Generator Network: Needs to generate data (signals, images)
- Discriminator Network: Distinguishes generated and real data



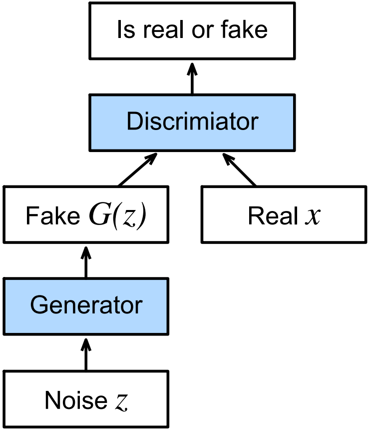
Generative Adversarial Network [Goodfellow et al., 2014]

- Training Process
  - Networks compete with each other
  - Generator attempts to fool the Discriminator
  - Discriminator adapts to the new fake/generated data
  - This information is used to improve the generator
- Discriminator:
  - A binary classifier, outputs a scalar prediction $o \in \mathbb{R}$
  - Applies *sigmoid* function to obtain predicted probability $D(\mathbf{x}) = 1/(1+e^{-o})$
  - The label y for true data is 1 and for fake data is 0
  - Train the discriminator to minimize the cross-entropy loss: $\min_D\{-y \log D(\mathbf{x}) - (1-y)\log(1-D(\mathbf{x}))\}$



Generative Adversarial Network [Goodfellow et al., 2014]

- For a given discriminator D, update the parameters of the generator G to maximize the cross-entropy loss when $y = 0$
  $\max_G\{-(1-y)\log(1-D(G(\mathbf{z})))\} = \max_G\{-\log(1-D(G(\mathbf{z})))\}$
- Conventionally, we minimize
  $\min_G\{-y\log(D(G(\mathbf{z})))\} = \min_G\{-\log(D(G(\mathbf{z})))\}$
- D and G are playing a "minimax" game with the comprehensive objective function
  $min_D max_G\{-E_{x\sim\text{Data}} log D(\mathbf{x}) - E_{z\sim\text{Noise}} log(1 - D(G(\mathbf{z})))\}$



Complex NN

- Complex Convolution: $\mathbf{W} * \mathbf{h} = (\mathbf{A} * \mathbf{x} - \mathbf{B} * \mathbf{y}) + i(\mathbf{B} * \mathbf{x} + \mathbf{A} * \mathbf{y})$
- ReLU:
  - ModReLU

$$\mathrm{modReLU}(z) = \mathrm{ReLU}(|z| + b)e^{i\theta_z} = \begin{cases} (|z| + b)\frac{z}{|z|} & \text{if } |z| + b \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

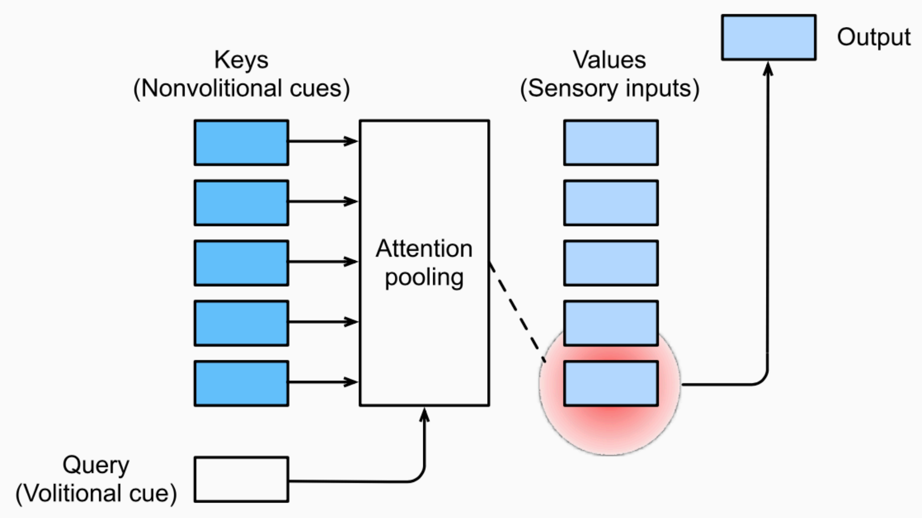
  - CReLU

$$\mathbb{C}\mathbb{R}\mathbb{L}\mathbb{U}(z) = \mathrm{ReLU}(\Re(z)) + i\,\mathrm{ReLU}(\Im(z))$$

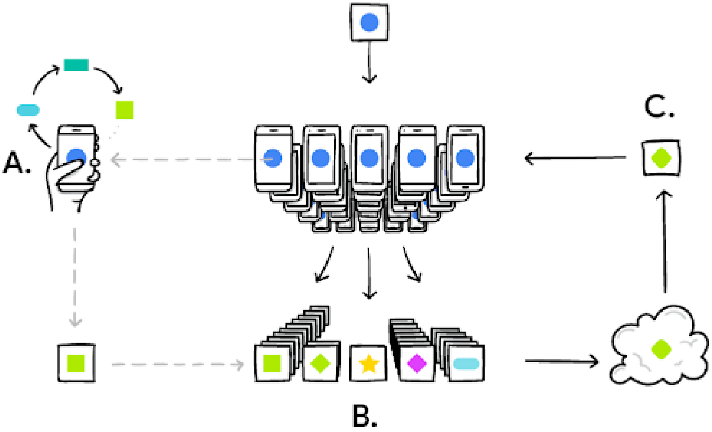
  - ZReLU

$$z\,\mathrm{ReLU}(z) = \begin{cases} z & \text{if } \theta_z \in [0, \pi/2] \\ 0 & \text{otherwise} \end{cases}$$

Attention Based NN / Transformer



Federated Learning



Wireless Applications

Wireless Applications

- Signal Detection
- Channel Encoding and Decoding
- Channel Estimation, Prediction, and Compression
- End-to-End Communications and Semantic Communications
- Distributed and Federated Learning and Communications
- Resource Allocation
- RF Fingerprinting
- Federated Learning
- Waveform Generation

https://www.comsoc.org/publications/best-readings/machine-learning-communications