Gambling for Redemption and Self-Fulfilling Debt Crises*

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ABSTRACT

We develop a model for analyzing the sovereign debt crises of 2010–2013 in the Eurozone. The government sets its expenditure-debt policy optimally. The need to sell large quantities of bonds every period leaves the government vulnerable to self-fulfilling crises in which investors, anticipating a crisis, are unwilling to buy the bonds, thereby provoking the crisis. In this situation, the optimal policy of the government is to reduce its debt to a level where crises are not possible. If, however, the economy is in a recession where there is a positive probability of recovery in fiscal revenues, the government may optimally choose to “gamble for redemption,” running deficits and increasing its debt, thereby increasing its vulnerability to crises.

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1. Introduction

This paper develops a model of the sorts of sovereign debt crisis that we have witnessed during 2010–2013 in such European countries as Greece, Ireland, Italy, Portugal, and Spain. The government sets its expenditure-debt policy optimally, given a probability of a recovery in fiscal revenues. In doing so, the government can optimally choose to “gamble for redemption,” and the economy can be optimally driven into a situation of increasing vulnerability to speculative attacks in the form of self-fulfilling debt crises. We provide a theory of sovereign debt crises in which both borrowers and lenders behave optimally, but where countries borrow so much, and lenders are willing to lend them that much, as to make a default unavoidable. Our theory contrasts alternative explanations based on misperceptions or other forms of irrationality.

Our paper provides a benchmark for analyzing government behavior in the face of fiscal pressures and a tool for testing the implications of alternative policy responses. In our analysis, as in Cole and Kehoe (1996, 2000), we characterize in a simple Markov structure the time consistent policy of a strategic government that is faced with non-strategic bond holders.

![Figure 1: Government debt in selected European countries](image)

The recent worldwide recession — which continues in many countries — and the policies intended to overcome it have generated very large government budget deficits and increases in government debt over the entire developed world. Figure 1 plots debt to GDP ratios for the most
troubled European economies — the so-called PIIGS, Portugal, Ireland, Italy, Greece, Spain — as well as Germany. These data are at odds with the theory developed by Cole and Kehoe (1996, 2000), who argue that the optimal policy of a government that faces a positive probability of a self-fulfilling debt crisis is to pay down its debt.

Figure 2: Real GDP in selected Eurozone countries

In our model, the crucial element that drives a government to risk suffering a self-fulfilling debt crisis is the drop in government revenues that occurs as the result of a recession in the private sector of the economy. Figure 2 shows that the worldwide recession that started in 2008 is still ongoing in Greece, Ireland, Italy, Portugal, and Spain. Notice that the drops in real government revenues (deflated by the GDP deflator) — presented in figure 3 — are also very large.

Our analysis shows that — under certain conditions, which correspond to parameter values and the fundamentals of the economy — it is optimal for the government to “gamble for redemption.” By this we mean that the government does not undertake painful adjustments to reduce spending, hoping for a recovery of government revenues, and debt continues to increase. Indeed, the government strategy follows a martingale gambling strategy that will send the
economy into the crisis zone if the recovery does not happen soon enough. Under other conditions, however, the government gradually reduces the level of debt to exit the crisis zone and avert the possibility of a liquidity crisis, as in Cole and Kehoe (1996, 2000). The data in figures 4 and 5 indicate that the governments of the GIIPS continued to borrow even as the spreads on their debt indicated the danger of self-fulfilling debts crises.

![Figure 3: Real government tax revenues in selected Eurozone countries](image)

In our environment not running down debt (or running it up until default is unavoidable) can be part of the optimal strategy under some circumstances. In contrast, Reinhart and Rogoff (2009) argue that the reason some countries fail to adjust and are vulnerable to a potential crisis is because somehow both the governments and their lenders are fooling themselves into thinking that “this time is different.” As such, a country’s vulnerability to a crisis would be the result of self-delusion and lack of rationality. In contrast to their view, we provide a model in which such apparently irrational behavior can be an optimal response to fundamentals by both borrowing governments and lenders that perfectly understand the risks of a crisis.

This paper is most related to those of Cole and Kehoe (1996, 2000). Similar frameworks have been used to analyze currency crisis following Calvo (1988). Cole and Kehoe provide a dynamic stochastic general equilibrium model of a country subject to the possibility of a self-fulfilling debt crisis in every period. The substantial difference between their framework and
ours is that, in their framework, debt crises are liquidity crises that are due solely to the inability to roll over debt. As such, a decisive action by a third party providing a loan or a bailout would be enough to avert the problem. Indeed, Cole and Kehoe (1996, 2000) intended their model to shed light on the financial troubles of Mexico in 1994–95, and, in that occasion, the decisive action of the Clinton administration on 31 January 1995 was enough to avert the crisis.

European Union rescue packages for countries Greece, Ireland, and Portugal have not had the same healing properties. In fact, quite the opposite seems to be the case, with the spreads on bonds, relative to Germany’s, continuing to rise despite the announcements and first implementations of the rescue packages. This indicates more fundamental solvency problems than those present in a standard liquidity crisis of the type studied in Cole and Kehoe, as discussed by Chamley and Pinto (2011) for the Greek case. Our model accommodates this issue.

Figure 4: Net government borrowing in selected Eurozone countries
Note: Net government borrowing in Ireland in 2010 was 30.9 percent of GDP.

The model we propose extends Cole-Kehoe to incorporate into the analysis a severe recession of uncertain recovery. By doing that, we are incorporating a motive for consumption smoothing in the tradition of Arellano (2008) and Aguiar and Gopinath (2007), who focus on default incentives on international borrowing over the cycle, but do not allow for self-fulfilling
debt crises. For studies of individual (rather than sovereign) default of unsecured debt, see Chaterjee et al. (2007). In our model there is a well defined trade-off between the benefits of consumption smoothing and the increased vulnerability associated to increasing the level of debt. Our quantitative results relate this trade-off (and whether we should observe "gambling for redemption" as an optimal response) to fundamentals of the economy such as the severity of a recession, the likelihood of a recovery, the existing stock of debt, etc.

Our model establishes conditions under which a debt crisis can occur, and how that possibility shapes optimal government’s behavior, but is silent about why at a particular point in time a crisis might or might not occur. Indeed, once the government is in the crisis zone (and we show under what conditions a government will find it optimal to enter it) a potential crisis is triggered by a non-fundamental random variable: a sunspot. A very interesting strand of the literature, starting with the work of Morris and Shin (1998), would relate the probability of such an event happening to lack of perfect knowledge within a global game framework. While this strand of the literature focuses on why a crisis might happen at a particular point in time, we focus on the conditions under which a benevolent government might find it optimal to subject itself to such a vulnerable position.

Figure 5: Yields on five-year government bonds in selected Eurozone countries
Note: Yields on Greek bonds were 26.8 percent per year in 2011:3.

We first analyze theoretically a model with one period bonds for special cases. Later we will solve computationally the whole model for longer-lived bonds. Gambling for redemption is optimal because the recession is very severe (as seen in the data for the GIIPS) and we assume there is a non-trivial probability of a recovery. We will also discuss the implications of the existence of bonds at different maturities. The basic implication is that as maturity increases the incentives to gamble increase, even for intermediate levels of debt.

2. General model

The model has a similar structure to that of Cole and Kehoe (1996, 2000). The major innovation is that output is stochastic, introducing a motive for consumption smoothing in the Aiyagari-Huggett tradition (i.e. there is uninsurable idiosyncratic risk), making it sometimes optimal for the government to gamble for redemption. With this added complication, we have chosen to simplify the model by eliminating the representative household’s consumption-investment choice. It would be conceptually straight-forward, but tedious, to allow for private investment.

The state of the economy in every period $s = (B, a, z_{-1}, \zeta)$ is the level of government debt $B$, whether or not the private sector is in normal conditions $a = 1$ or in a recession $a = 0$, whether or not default has occurred in the past $z_{-1}$, and the value of the sunspot variable $\zeta$. The country’s GDP is

$$y(a,z) = A^{1-a}Z^{1-z}\bar{y}$$

(1)

where $1 > A, Z > 0$. Before period 0, $a = 1$, $z = 1$. In period 0, $a$ unexpectedly becomes $a_0 = 0$ and GDP drops from $y = \bar{y}$ to $y = A\bar{y} < \bar{y}$. In every period $t$, $t = 1, 2, ..., a_t$ becomes 1 with probability $p$, $1 > p > 0$. Once $a = 1$, it stays equal to 1 forever. The drop in productivity by the factor $Z$ is the country’s default penalty. Once $z = 0$, it stay equal to 0 forever. Here the default penalty occurs in the same period as the crisis. Figure XX illustrates a possible evolution of the country’s GDP over time. In terms of the crises in the Eurozone, we can think of $t = 0$ as 2008.
Government tax revenue is \( \theta y(a,z) \) where we assume, as do Cole and Kehoe (1996, 2000) to keep things simple, that the tax rate \( \theta \) is fixed. Given that there is no consumption-investment choice, the consumption of the representative household is

\[
c(a,z) = (1-\theta)y(a,z)\quad \tag{2}
\]

The government offers \( B' \) in new bonds for sale and chooses whether to repay or nor the debt becoming due, \( B \). The government’s budget constraint is

\[
g + zB = \theta y(a,z) + q(B',s)B', \quad \tag{3}
\]

where \( q(B',s) \) is the price that international bankers that pay for \( B' \).

In every period, \( \zeta \) is drawn from the uniform distribution on \([0,1]\). If \( \zeta > 1-\pi \), international bankers expect there to be a crisis and do not lend to the government if such a crisis would be self-fulfilling. This allows us to set the probability of a self-fulfilling crisis at an arbitrary level \( \pi \), \( 1 \geq \pi \geq 0 \), if the level of debt is high enough for such a crisis to be possible.

The timing within each period is like that in Cole and Kehoe (1996, 2000):

1. The shocks \( a \) and \( \zeta \) are realized, the aggregate state is \( s = (B,a,z_{-1},\zeta) \), and the government chooses how much debt \( B' \) to sell.

2. Each of a continuum of measure one of international bankers chooses how much debt \( b' \) to purchase. (In equilibrium, of course, \( b' = B' \).)

3. The government makes its default decision \( z \), which determines \( y, c \), and \( g \).

The crucial elements of this timing are: First, the government faces a time-consistency problem because, when offering \( B' \) for sale, it cannot commit to repay \( B \). Second, since all uncertainty has been resolves at the beginning of the period, there is perfect foresight in equilibrium within the period and, in particular, international bankers do not lend if they know the government will default. Third, whether or not a crisis occurs during the period depends on \( B \) while — if no crisis occurs — the price of new bonds depend on \( B' \).

Given this timing the government’s problem reduces to choosing \( c,g,B',z \) to solve

\[
V(s) = \max\ u(c,g) + \beta EV(s')\quad \tag{4}
\]
\[ \text{s.t. } c = (1 - \theta) y(a, z) \]
\[ g + zB = \theta y(a, z) + q(B', s)B' \]
\[ z = 0 \text{ if } z_{-1} = 0. \]

Here \( z = 1 \) is the decision not to default, and \( z = 0 \) is the decision to default.

In general, we assume that, for any \( b \) such that \( Ay - b \) is an element of the feasible set of levels for government expenditures \( g \),
\[ u_g((1 - \theta)A\bar{y}, \theta A\bar{y} - b) > u_g((1 - \theta)\bar{y}, \theta \bar{y} - b). \]

This assumption provides the government with the incentive to transfer resources into the current period during a recession from future periods in which the economy has recovered. It is satisfied by any concave utility function separable in \( c \) and \( g \). It is also satisfied by functions like \( \log(c + g - \bar{c} - \bar{g}) \).

2.1. Bond prices

International bankers are risk neutral with discount factor \( \beta \) so that the bond prices \( q(B', s) \) are determined by the probability of default in the next period. There is a continuum of measure one of bankers. Each solves the dynamic programming problem
\[ W(b, B', s) = \max x + \beta EW(b', B'', s') \]
\[ x + q(B', s)b' = w + z(B', s, q(B', s))b \]
\[ x \geq 0, \ b \geq -A. \]

The constraint \( b \geq -A \) eliminates Ponzi schemes, but \( A \) is large enough so that the constraint does not otherwise bind. We assume that the banker’s endowment of consumption good \( w \) is large enough to rule out corner solutions in equilibrium. We refer to this assumption as the assumptions that the banker has deep pockets.

There are four cutoff levels of debt: \( \bar{b}(a), \ \bar{B}(a), \ a = 0,1 \):

1. If \( B \leq \bar{b}(0) \), the government does not default when the private sector is in a recession even if international bankers do not lend, and, if \( B > \bar{b}(0) \), the government defaults when the private sector is in a recession if international bankers do not lend.
2. If \( B \leq b(1) \), the government does not default when the private sector is in normal conditions even if international bankers do not lend, and, if \( B > b(1) \), the government defaults when the private sector is in normal conditions if international bankers do not lend.

3. If \( B \leq B(0) \), the government does not default when the private sector is in a recession if international bankers lend, and, if \( B > B(0) \), the government defaults when the private sector is in a recession even if international bankers lend.

4. If \( B \leq B(1) \), the government does not default when the private sector is in normal conditions if international bankers lend, and, if \( B > B(1) \), the government defaults when the private sector is in normal conditions even if international bankers lend.

The assumption that once \( z = 0 \), it stays equal to 0 forever says that a country that defaults is permanently excluded from international borrowing or lending. This assumption can be modified at the cost of complicating the analysis. The assumption has two consequences for the relation of the bond price \( q \) to the current state \( s \): First, once default has occurred, international bankers do not lend:

\[
q(B', (B, a, 0, \zeta)) = 0.
\]

Second, during a crisis, international bankers do not lend:

\[
q(B', (B, a, 1, \zeta)) = 0
\]

whenever \( B > \bar{b}(a) \) and \( \zeta > 1 - \pi \). Otherwise, the bond price \( q \) only depends on the amount of bonds \( B' \) that the government offers for sale.

We will show that \( \bar{b}(0) < \bar{b}(1) \), \( \bar{b}(0) < \bar{B}(0) \), \( \bar{b}(1) < \bar{B}(1) \), and \( \bar{B}(0) < \bar{B}(1) \). Here, we first consider the case where \( \bar{b}(1) < \bar{B}(0) \), that is, where

\[
\bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1).
\]

The first order condition for to the international bankers’ utility maximization problem implies that

\[
q(B', s) = \beta E_z(B'(s'), s', q(B'(s'), s')) ,
\]
which implies that in recession times:

\[
q(B', (B, 0, 1, \zeta)) = \begin{cases} 
\beta & \text{if } B' \leq \bar{b}(0) \\
\beta \left( p + (1 - p)(1 - \pi) \right) & \text{if } \bar{b}(0) < B' \leq \bar{b}(1) \\
\beta(1 - \pi) & \text{if } b(1) < B' \leq B(0) \\
\beta p(1 - \pi) & \text{if } B(0) < B' \leq B(1) \\
0 & \text{if } B(1) < B'
\end{cases}
\]

and in normal times:

\[
q(B', (B, 1, 1, \zeta)) = \begin{cases} 
\beta & \text{if } B' \leq b(1) \\
\beta(1 - \pi) & \text{if } b(1) < B' \leq B(1) \\
0 & \text{if } B(1) < B'
\end{cases}
\]

Figure 6: Bond prices as a function of new bonds offered and state of private sector

Now consider the case where \( \bar{B}(0) < \bar{b}(1) \), that is, where

\[
\bar{b}(0) < \bar{B}(0) < \bar{b}(1) < \bar{B}(1).
\]

Here the solution to the international bankers’ utility maximization problem implies that
There is yet a third case where

\[ b(0) < b(1) = B(0) < B(1). \]

Here

\[ q(B', (B, 0, 1, \zeta)) = \begin{cases} 
\beta & \text{if } B' \leq \overline{b}(0) \\
\beta (p + (1 - p)(1 - \pi)) & \text{if } \overline{b}(0) < B' \leq \overline{B}(0) \\
\beta p & \text{if } \overline{B}(0) < B' \leq \overline{b}(1) \\
\beta p(1 - \pi) & \text{if } \overline{b}(1) < B' \leq \overline{B}(1) \\
0 & \text{if } \overline{B}(1) < B'. 
\end{cases} \]

Since the second and the third cases, where \( \overline{B}(0) \leq \overline{b}(1) \), are only possible for very low values of \( A \) (catastrophic recessions), we will focus on the first case. Besides, in those other cases the model is not very informative about self-fulfilling crisis since after the recession hits the country is either not vulnerable to crisis or defaults directly.

### 2.2. Definition of equilibrium

An equilibrium is a value function for government \( V(s) \) and policy functions \( B'(s) \) and \( z(B', s, q) \) and \( g(B', s, q) \), a value function for bankers \( W(b, B', s) \) and policy correspondence \( b'(b, B', s) \), and a bond price function \( q(B', s) \) such that

1. \( V(s) \) and \( B'(s) \) solve the government’s problem at the beginning of the period:

\[
V(B, a, z, z', \zeta') = \max \ u(c, g) + \beta EV(B', a', z, \zeta') \\
\text{s.t.} \ c = (1 - \theta) y(a, z(B', s, q(B', s))) \\
g(B', s, q(B', s)) + z(B', s, q(B', s)) B = \theta y(a, z) + q(B', s) B'.
\]
2. \( b'(b,B',s) \) solves the banker’s problem and \( q(B',s) \) is consistent with market clearing and rational expectations:

\[
B'(s) \in b'(b,B',s) \\
q(B',s) = \beta Ez(B',s,q(B',s)).
\]

3. \( z(B',s,q) \) and \( g(B',s,q) \) solve the government’s problem at the end of the period:

\[
\max u(c,g) + \beta EV(B',a',z,\zeta') \\
\text{s.t. } c = (1-\theta)y(a,z) \\
g + zB = \theta y(a,z) + qB' \\
z = 0 \text{ or } z = 1, \text{ but } z = 0 \text{ if } z_{-1} = 0.
\]

Notice that, when the government solves its problem at the beginning of the period, it takes as given the optimal responses both of international bankers and itself later in the period. In particular, the government cannot commit to repaying its debt and not defaulting later in the period. Furthermore, since the occurrence of a crisis depends on the amount of debt to be repaid \( B \), not the amount of debt offered for sale \( B' \), once a sunspot has occurred that signals that a self-fulfilling crisis will take place that period, there is nothing that the government can do to avoid it.

It is also worth noting that this model has many equilibria. Our definition of equilibrium restricts our attention to equilibria with a simple Markov structure. Many other possibilities are possible. If we include the date in the state \( s = (B,a,z_{-1},\zeta,t) \), for example, we could allow crises to occur only in even periods \( t \) or in periods that are prime numbers, or we could allow the probability of a crisis \( \pi \) to be time varying in other ways. The advantage of our simple Markov structure is that it makes it easy to characterize and compute equilibria.

3. **Self-fulfilling liquidity crises**

Although we are able to prove some propositions about the equilibria of the general model, we need to resort to numerical examples to illustrate the possibilities and to do comparative statics analysis. Before turning to the results for the general model, we study two special cases, where we can provide analytical characterizations of the equilibria in which we are
interested. The first is the case where \( a = 1 \), that is, where the private sector has recovered and where there is no incentive for the government to gamble for redemption. This is a simplified version of the model of Cole-Kehoe (1996, 2000) without private capital. Notice that we can easily modify the analysis of this case to study the limiting case where \( a = 0 \) and \( p = 0 \), that is, where there is a recession but no possibility for recovery, simply by replacing \( y \) with \( Ay \) in what follows. In this case, where self-fulfilling crises are possible, but where there is no incentive for the government to gamble for redemption, the optimal strategies of the government involve either leaving debt constant or running it down to eliminate the possibility of a crisis. In the next section, we consider the other extreme case, where recovery is possible but not self-fulfilling crises.

We start by assuming that \( \pi = 0 \). Notice that since a recovery has already occurred in the private sector \( p \) is irrelevant. To derive the optimal government policy, we solve

\[
\max \sum_{t=1}^{\infty} \beta^t u(c_t, g_t)
\]

s.t.
\[
c_t = (1 - \theta)\bar{y}
\]
\[
g_t + B_t = \theta\bar{y} + \beta B_{t+1}
\]
\[
B_t = B
\]
\[
B_t \leq \bar{B}(1).
\]

The first order conditions are

\[
\beta^t u_g((1 - \theta)\bar{y}, g_t) = \lambda_t
\]

\[
\lambda_{t+1} = \beta \lambda_t.
\]

The transversality condition is

\[
\lim_{t \to \infty} \lambda_t B_{t+1} \geq 0.
\]

The first order conditions imply that

\[
u_g((1 - \theta)\bar{y}, g_t) = u_g((1 - \theta)\bar{y}, g_{t-1})
\]

\[
g_{t+1} = g_t = \hat{g}.
\]
Consequently,

\[ B_{t+1} = \frac{1}{\beta}(\hat{g} + B_t - \theta \bar{y}). \]

Suppose that

\[ \hat{g} = \theta \bar{y} - (1 - \beta)B. \]

Then \( B_t = B \). Otherwise \( B_t \) is explosive. It is easy to show that too low a \( \hat{g} \) results in a path for \( B_t \) that violates the transversality condition. Too high a \( \hat{g} \) results in a path for \( B_t \) that hits \( B(1) \). It is easy to prove that this cannot be optimal.

We can calculate the value of being in state \( s = (B,a,z_{-1},\zeta) = (B,1,1,\zeta) \) as

\[ V(B,1,1,\zeta) = \frac{u((1 - \theta) \bar{y}, \theta \bar{y} - (1 - \beta)B)}{1 - \beta}. \]

The calculation of utility when default has occurred, when \( z = 0 \), is mechanical. In this case \( B = 0 \) and

\[ c = (1 - \theta)Z\bar{y}, \quad g = \theta Z\bar{y}. \]

Notice that, once a default has occurred, \( \zeta \) and \( \pi \) are irrelevant. Consequently, when \( s = (B,a,z_{-1},\zeta) = (B,1,0,\zeta) \)

\[ V(B,1,0,\zeta) = V_d(1) = \frac{u((1 - \theta)Z\bar{y}, \theta Z\bar{y})}{1 - \beta}, \]

Let us calculate \( \tilde{b}(1) \). Let \( V_n(B,a,q) \) be the value of not defaulting. The utility of repaying \( B \) even if the international bankers do not lend is

\[ V_n(B,1,0) = u((1 - \theta) \bar{y}, \theta \bar{y} - B) + \frac{\beta u((1 - \theta) \bar{y}, \theta \bar{y})}{1 - \beta}, \]

while the utility of defaulting \( V_d(a) \) is
$$V_d(1) = \frac{u((1-\theta)Zy,\theta Zy)}{1-\beta}.$$  

Consequently, $b(1)$ is determined by the equation

$$V_n(b(1),1,0) = V_d(1)$$

$$u((1-\theta)y,\theta y - b(1)) + \frac{\beta u((1-\theta)y,\theta y)}{1-\beta} = u((1-\theta)Zy,\theta Zy)$$

$$u((1-\theta)Zy,\theta Zy) - u((1-\theta)y,\theta y - b(1)) = \frac{\beta}{1-\beta} \left( u((1-\theta)y,\theta y) - u((1-\theta)Zy,\theta Zy) \right).$$

Let us now characterize $B(1)$. We first find the stationary upper limit on debt $B(1)$ and then show that $B(1) = B^*(1)$. Suppose that international lenders set $q = \beta$ when $a = 1$. Then, if the government decides to repay its debt

$$V_n(B,1,\beta) = \frac{u((1-\theta)y,\theta y - (1-\beta)B)}{1-\beta},$$

while the value of defaulting is

$$V_d(B,1,\beta) = u((1-\theta)Zy,\theta Zy + \beta B) + \frac{\beta u((1-\theta)Zy,\theta Zy)}{1-\beta}.$$  

Consequently, $B^*(1)$ is determined by the equation

$$V_n(B^*(1),1,\beta) = V_d(B^*(1),1,\beta)$$

$$\frac{u((1-\theta)y,\theta y - (1-\beta)B^*(1))}{1-\beta} = u((1-\theta)Zy,\theta Zy + \beta B^*(1)) + \frac{\beta u((1-\theta)Zy,\theta Zy)}{1-\beta}$$

$$u((1-\theta)Zy,\theta Zy + \beta B^*(1)) - u((1-\theta)y,\theta y -(1-\beta)B^*(1)) = \frac{\beta}{1-\beta} \left( u((1-\theta)y,\theta y - (1-\beta)B^*(1)) - u((1-\theta)Zy,\theta Zy) \right).$$

Suppose that, at $t = 0$, the government finds itself with $B_0 > B^*(1)$. Then it cannot be optimal to run down its debt to $B_1 \leq B^*(1)$ in one period, setting
\[ g_0 = \theta y - B_0 + \beta B_1 \]

in \( t = 0 \) and

\[ g_t = \theta y - (1 - \beta) B_1 \]

thereafter. To demonstrate this, suppose, to the contrary, that

\[
u((1 - \theta) y, \theta y - B_0 + \beta B_1) + \frac{\beta u((1 - \theta) y, \theta y - (1 - \beta) B_0)}{1 - \beta} \geq u((1 - \theta) Z y, \theta Z y + \beta B_0) + \frac{\beta u((1 - \theta) Z y, \theta Z y)}{1 - \beta} \]

If we choose \( B_1 \) to maximize the left hand side of this inequality, the utility of not defaulting, we set \( B_1 = B_0 \). This implies that

\[
u((1 - \theta) y, \theta y - B_0 + \beta B_0) + \frac{\beta u((1 - \theta) y, \theta y - (1 - \beta) B_0)}{1 - \beta} \geq u((1 - \theta) y, \theta y - B_0 + \beta B_1) + \frac{\beta u((1 - \theta) y, \theta y)}{1 - \beta} \]

for any \( B_1 \neq B_0 \). This, however, contradicts or assumption that \( B_0 > \bar{B}^*(1) \).

Let us now consider the case where \( \pi > 0 \). First, observe that the lower bound \( \bar{b}(1) \) is independent of \( \pi \).

Suppose that \( B_0 > \bar{b}(1) \) and the government decides to reduce \( B \) to \( \bar{b}(1) \) in \( T \) periods, \( T = 1, 2, \ldots, \infty \). The first order conditions for the government’s problem imply that

\[ g_t = g^T(B_0; \pi). \]

The government’s budget constraints are

\[
g^T(B_0; \pi) + B_0 = \theta \bar{y} + \beta(1 - \pi) B_1 \\
g^T(B_0; \pi) + B_1 = \theta \bar{y} + \beta(1 - \pi) B_2 \\
\vdots \\
g^T(B_0; \pi) + B_{T-2} = \theta \bar{y} + \beta(1 - \pi) B_{T-1} \\
g^T(B_0; \pi) + B_{T-1} = \theta \bar{y} + \beta \bar{b}(1; p, \pi). \]

Multiply each equation by \((\beta(1-\pi))^t\) and adding, we obtain

\[
\sum_{t=0}^{T-1} (\beta(1-\pi))^t g^T(B_0;\pi) + B_0 = \sum_{t=0}^{T-1} (\beta(1-\pi))^t \theta y + (\beta(1-\pi))^{T-1} \beta b(1)
\]

\[
g^T(B_0;\pi) = \theta y - \frac{1 - \beta(1-\pi)}{1 - (\beta(1-\pi))^T} (B_0 - (\beta(1-\pi))^{T-1} \beta b(1; p, \pi)).
\]

Notice that

\[
g^\infty(B_0;\pi) = \lim_{T \to \infty} g^T(B_0;\pi) = \theta y - (1 - \beta(1-\pi))B_0.
\]

We can compute the value \(V^T(B_0;\pi)\) of each of the policies of running down the debt in \(T\) periods, \(T = 1, 2, ..., \infty\). Letting \(V^T_t(B_0;\pi)\) be the value of the policy where there are still \(t\) periods to go in running debt, we can write

\[
V^T_1(B_0;\pi) = u((1-\theta)\overline{y}, g^T(B_0;\pi)) + \beta(1-\pi)V^T_0(B_0;\pi) + \frac{\beta \pi u((1-\theta)\overline{y}, \theta\overline{y})}{1 - \beta}
\]

\[
V^T_2(B_0;\pi) = u((1-\theta)\overline{y}, g^T(B_0;\pi)) + \beta(1-\pi)V^T_1(B_0;\pi) + \frac{\beta \pi u((1-\theta)\overline{y}, \theta\overline{y})}{1 - \beta}
\]

\[
V^T_T(B_0;\pi) = u((1-\theta)\overline{y}, g^T(B_0;\pi)) + \beta(1-\pi)V^T_{T-1}(B_0;\pi) + \frac{\beta \pi u((1-\theta)\overline{y}, \theta\overline{y})}{1 - \beta}
\]

Notice that \(g\) increases from \(g^T(B_0;\pi)\) to \(\theta\overline{y}\) in period \(T\). To calculate \(V^T(B_0;\pi)\), we use backwards induction

\[
V^T_2(B_0;\pi) = (1 + \beta(1-\pi))u((1-\theta)\overline{y}, g^T(B_0;\pi)) + \frac{\beta \pi u((1-\theta)\overline{y}, \theta\overline{y})}{1 - \beta} + \beta(1-\pi) \frac{\beta u((1-\theta)\overline{y}, \theta\overline{y})}{1 - \beta}
\]
\[ V_T^T(B_0; \pi) = (1 + \beta(1 - \pi) + (\beta(1 - \pi))^2)u((1 - \theta)y, g^T(B_0; \pi)) \]
\[ + (1 + \beta(1 - \pi))^{\beta \pi u((1 - \theta)Zy, \theta Zy)} \]
\[ + (\beta(1 - \pi))^{2} \beta\pi u((1 - \theta)y, \theta y) \]
\[ V_T^T(B_0; \pi) = (1 + \beta(1 - \pi) + (\beta(1 - \pi))^2 + ... + (\beta(1 - \pi))^{T-1})u((1 - \theta)y, g^T(B_0; \pi)) \]
\[ + (1 + \beta(1 - \pi) + (\beta(1 - \pi))^2 + ... + (\beta(1 - \pi))^{T-2}) \beta\pi u((1 - \theta)Zy, \theta Zy) \]
\[ + (\beta(1 - \pi))^{T-2} \beta\pi u((1 - \theta)y, \theta y) \]

and, of course, \( V_T^T(B_0; \pi) = V_T^T(B_0; \pi) : \)
\[ V^T(B_0; \pi) = \frac{1 - (\beta(1 - \pi))^T u((1 - \theta)y, g^T(B_0; \pi))}{1 + \beta(1 - \pi)} + \frac{1 - (\beta(1 - \pi))^{T-1} \beta\pi u((1 - \theta)Zy, \theta Zy)}{1 + \beta(1 - \pi)} \]
\[ + (\beta(1 - \pi))^{T-2} \beta\pi u((1 - \theta)y, g^T(B_0; \pi)) \]

Notice that
\[ V^\infty(B_0; \pi) = \frac{u((1 - \theta)\overline{y}, \theta \overline{y} - (1 - \beta(1 - \pi))B_0)}{1 + \beta(1 - \pi)} + \frac{\beta\pi u((1 - \theta)\overline{y}, \theta \overline{y})}{(1 - \beta)(1 + \beta(1 - \pi))}. \]

To find \( B(1) \), we solve
\[ \max \left[ V^1(B(1; p, \pi)), V^2(B(1; p, \pi)), ..., V^\infty(B(1; p, \pi); \pi) \right] \]
\[ = u((1 - \theta)\overline{y}, \theta \overline{y} + (1 - \pi)B(1; p, \pi)) + \frac{\beta\pi u((1 - \theta)\overline{y}, \theta \overline{y})}{1 - \beta}. \]

Our arguments have produced the following analytical characterization of \( V(B,1,1,\zeta) : \)
$V(B,1,1,\zeta) = \begin{cases} 
\frac{u((1-\theta)y,\theta y)}{1-\beta} & \text{if } B \leq b(1) \\
\max\left[V^1(B;\pi), V^2(B;\pi), \ldots, V^\infty(B;\pi)\right] & \text{if } b(1) < B \leq B(1), \zeta \leq 1-\pi \\
u((1-\theta)Zy,\theta Zy) & \text{if } b(1) < B \leq B(1), 1-\pi < \zeta \\
u((1-\theta)Zy,\theta Zy) & \text{if } B(1) < B 
\end{cases}$

Some of the different possibilities for optimal government strategies — which vary with the initial debt — are illustrated in figure 5.

Figure 5: Optimal debt policy with self-fulfilling crises

4. Consumption smoothing without self-fulfilling crises

Suppose now that $a = 0$ and $\pi = 0$. That is, no self-fulfilling crises are possible, but the private sector is in a recession and faces the probability $p$, $1 > p > 0$, of recovering in every period. We can also interpret this as the limiting case in which crises can occur, but the government and the international bankers assign probability $\pi = 0$ to them.
In this section, we argue that the optimal government policies is to increase its debt as long as \( a = 0 \). In fact, if the country is unlucky in the sense that \( a = 0 \) long enough, the government may choose to eventually default. Consequently, the upper limits on the debt, \( \bar{B}(0; p, 0) \) and \( \bar{B}(1; p, 0) \) are crucial for our analysis. Notice that, since

\[
\bar{B}(1; p, 0) = \bar{B}(1; 0, 0),
\]

because the probability of recovery does not matter once it has occurred, we have already derived a condition that determines \( \bar{B}(1; p, 0) \).

\[
\begin{align*}
&u((1-\theta)Z\theta, \theta\bar{y} + \beta\bar{B}(1; p, 0)) - u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{B}(1; p, 0)) \\
&= \frac{\beta}{1-\beta} \left(u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{B}(1; p, 0)) - u((1-\theta)Z\theta, \theta\bar{y})\right).
\end{align*}
\]

To determine \( \bar{B}(0; p, 0) \), we suppose that the government runs up its debt to some level \( B \leq \bar{B}(0; p, 0) \), borrows \( \bar{B}(0; p, 0) < B' \leq \bar{B}(1; p, 0) \) at price \( \beta p \), then repays the next period if the private sector recovers and on defaults otherwise. The value of borrowing \( \bar{B}(1; p, 0) \) at price...
\( \beta p \), repaying the current debt, and then repaying in the next period if the private sector recovers and on defaulting otherwise is

\[
V_n(B,0,1;p,0) = u((1-\theta)Ay,\theta Ay + \beta pB(1;p,0) - B) \\
+ \beta(1-p) \left\{ \frac{u((1-\theta)Az,\theta Az) + \beta pu((1-\theta)\bar{Z}y,\theta \bar{Z}y)}{1-\beta(1-p)} \right\} \\
+ \frac{\beta p}{1-\beta} u((1-\theta)y,\theta y - (1-\beta)B(1;p,0))
\]

The value of borrowing \( B(1;p,0) \) at price \( \beta p \) and then defaulting is

\[
V_d(B,0,1;p,0) = u((1-\theta)Az,\theta Az + \beta pB(1;p,0)) \\
+ \frac{\beta(1-p)}{1-\beta} \left\{ \frac{u((1-\theta)\bar{Z}y,\theta \bar{Z}y) + \beta pu((1-\theta)\bar{Z}y,\theta \bar{Z}y)}{1-\beta(1-p)} \right\} \\
+ \frac{\beta p}{1-\beta} u((1-\theta)\bar{Z}y,\theta \bar{Z}y)
\]

The equation that determines \( \bar{B}(0;p,0) \) is, therefore,

\[
V_n(\bar{B}(0;p,0),0,1;p,0) = V_d(\bar{B}(0;p,0),0,1;p,0) \\
\]

\[
u((1-\theta)\bar{Y},\theta \bar{Y} + \beta p\bar{B}(1;p,0) - \bar{B}(0;p,0)) + \frac{\beta p}{1-\beta} u((1-\theta)\bar{Y},\theta \bar{Y} - (1-\beta)\bar{B}(1;p,0))
\]

\[
= u((1-\theta)\bar{Z},\theta \bar{Z} + \beta p\bar{B}(1;p,0)) + \frac{\beta p}{1-\beta} u((1-\theta)\bar{Z},\theta \bar{Z})
\]

The government may, in fact, choose a lower level of the debt than \( B(1;p,0) \), but \( \bar{B}(0;p,0) \) is the highest level of the debt \( B' \) at which the government can borrow at price \( q(B',s;p,\pi) = \beta \). Indeed, if the constraint \( B' \leq \bar{B}(1;p,0) \) does not bind, we can calculate the optimal \( B' \) by solving

\[
\max u((1-\theta)\bar{Y},\theta \bar{Y} + \beta pB' - B) \\
+ \beta(1-p) \left\{ \frac{u((1-\theta)\bar{Z},\theta \bar{Z}) + \beta pu((1-\theta)\bar{Z},\theta \bar{Z})}{1-\beta(1-p)} \right\} \\
+ \frac{\beta p}{1-\beta} u((1-\theta)\bar{Y},\theta \bar{Y} - (1-\beta)B')
\]

The first order condition is
Letting \( \hat{B}'(B) \) be the solution to this problem,

\[
B'(B) = \min[\hat{B}'(B), B(1; p, 0)].
\]

There are two cases:

1. The government chooses to never violate the constraint \( B \leq \bar{B}(0; p, 0) \), and the optimal debt converges to \( \bar{B}(0; p, 0) \) if \( a = 0 \) for sufficiently long.

2. The government chooses to default in \( T \) periods if \( a = 0 \) sufficiently long.

The crucial parameter in determining which of these two cases the economy is in is the default penalty factor \( Z \). If \( Z \) is sufficiently low, the government will choose to never default. If \( Z \) is close to 1, it will optimally choose to default after a sufficiently long number of periods in which \( a = 0 \).

4.1. Equilibrium with no default

Let us first consider the case where government chooses to never violate the constraint \( B \leq \bar{B}(0; p, 0) \). For this to be an equilibrium, two things must be true:

1. The expected discounted value of steady state utility at \( B = \bar{B}(0; p, 0) \) must be higher than that of defaulting after bankers have purchased \( B' = \bar{B}(0; p, 0) \) at price \( \beta \),

\[
u((1-\theta)\bar{A}y, \theta \bar{A}y - (1-\beta)\bar{B}(0; p, 0))
+ \beta(1-p)\left[u((1-\theta)\bar{A}y, \theta \bar{A}y - (1-\beta)\bar{B}(0; p, 0)) + \beta pu((1-\theta)\bar{A}y, \theta \bar{A}y - (1-\beta)\bar{B}(0; p, 0))\right]
+ \frac{\beta p}{1-\beta} u((1-\theta)\bar{A}y, \theta \bar{A}y - (1-\beta)\bar{B}(0; p, 0))
\geq u((1-\theta) AZy, \theta AZy + \beta \bar{B}(0; p, 0))
+ \beta(1-p)\left[u((1-\theta) AZy, \theta AZy) + \beta pu((1-\theta) AZy, \theta AZy)\right]
+ \frac{\beta p}{1-\beta} u((1-\theta) AZy, \theta AZy)
\]
The expected discounted value of steady state utility at $B = B(0; p, 0)$ must be higher than that of running up the debt one more time at price $\beta p$, repaying if the private sector recovers, and defaulting otherwise,

$$u((1-\theta)A\bar{y}, \theta A\bar{y} - (1-\beta)\bar{B}(0; p, 0)) + \beta(1-p)\left(\frac{u((1-\theta)A\bar{y}, \theta A\bar{y} - (1-\beta)\bar{B}(0; p, 0))}{1-\beta(1-p)} + \frac{\beta pu((1-\theta)\bar{y}, \theta \bar{y} - (1-\beta)\bar{B}(0; p, 0))}{(1-\beta)(1-\beta(1-p))}\right)$$

$$+ \frac{\beta p}{1-\beta} u((1-\theta)y, \theta y - (1-\beta)B(0; p, 0))$$

$$\geq u((1-\theta)Ay, \theta Ay + \beta pB'(B(0; p, 0)) - B(0; p, 0))$$

$$+ \beta(1-p)\left(\frac{u((1-\theta)AZy, \theta AZy)}{1-\beta(1-p)} + \frac{\beta pu((1-\theta)Zy, \theta Zy)}{1-\beta(1-\beta(1-p))}\right)$$

$$+ \frac{\beta p}{1-\beta} u((1-\theta)y, \theta y - (1-\beta)B'(B(0; p, 0)))$$

where $B'(B) = \min[\bar{B}'(B), \bar{B}(1; p, 0)]$.

If these two conditions are satisfied, the optimal government policy is the solution to the dynamic programming problem

$$V(B, a) = \max u((1-\theta)A^{1-a}\bar{y}, \theta A^{1-a}\bar{y} + \beta B' - B) + \beta EV(B', a')$$

s.t. $B \leq \bar{B}(0; p, 0)$

We write Bellman’s equation explicitly as

$$V(B, 0) = \max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta B' - B) + \beta(1-p)V(B', 0) + \beta pV(B', 1)$$

$$V(B, 1) = \max u((1-\theta)\bar{y}, \theta \bar{y} + \beta B' - B) + \beta V(B', 1).$$

The first order condition is

$$\beta u_y((1-\theta)A^{1-a}\bar{y}, \theta A^{1-a}\bar{y} + \beta B' - B) = \beta EV_B(B', a'),$$

while the envelope condition is

$$V_B(B, a) = -u_y((1-\theta)A^{1-a}\bar{y}, \theta A^{1-a}\bar{y} + \beta B' - B).$$
The envelope condition implies \( V(B, a) \) is decreasing in \( B \). A standard argument — that the operator on the space of functions defined by Bellman’s equations maps concave value functions into concave value functions — implies that \( V(B, a) \) is concave in \( B \).

Now, using the first order condition, we can use standard arguments to show that the policy function for debt \( B'(B, a) \) is increasing in \( B \) while the policy function for government spending \( g(B, a) \) is decreasing. Using our assumption that

\[
ug((1-\theta)A\bar{y}, \theta A\bar{y} - b) > ug((1-\theta)\bar{y}, \theta \bar{y} - b)
\]

we can argue that \( B'(0) > 0 \) and that it is impossible for \( B'(B) = B \) unless the constraint \( B \leq B(0; p, 0) \) binds, which implies that \( B'(B, 0) > B \). We have already argued that \( B'(B,1) = B \). Figure 7 illustrates some optimal government strategies as functions of the initial debt.

Figure 7: Optimal debt policy gambling for redemption when \( B \leq \bar{B}(0; p, 0) \) binds

4.2. Equilibrium with eventual default

Let us now consider the case where government chooses to violate the constraint \( B \leq \bar{B}(0; p, 0) \) with its sale of debt in period \( T \), defaulting in period \( T+1 \) unless the private
sector recovers. The optimal government policy along the branch of the uncertainty tree in which \( a_t = 0 \) is the solution to the finite horizon dynamic programming problem

\[
V_t(B_t,0) = \max u((1-\theta)A\bar{y},\theta A\bar{y} + \beta B_{t+1} - B_t) + \beta(1-p)V_{t+1}(B_{t+1},0) + \beta p \frac{u((1-\theta)\bar{y},\theta \bar{y} + (1-\beta)B_{t+1})}{1-\beta}
\]

\[
\text{s.t. } B(1;p,0) \geq B_{t+1} \geq B(0;p,0)
\]

\[
B_t \leq \bar{B}(0;p,0).
\]

We solve this by backwards induction with the terminal value function

\[
V_t(B_t,0) = \max u((1-\theta)A\bar{y},\theta A\bar{y} + \beta pB_{t+1} - B_t)
\]

\[
+ \beta(1-p) \frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta} + \beta p \frac{u((1-\theta)\bar{y},\theta \bar{y} + (1-\beta)B_{t+1})}{1-\beta}
\]

\[
\text{s.t. } B_{t+1} \leq \bar{B}(1;p,0)
\]

We then choose the value of \( T \) for which \( V_0(B_0,0) \) is maximal. As long the constraint \( B_{t+1} \geq \bar{B}(0;p,0) \) binds, we can increase the value of \( V_0(B_0,0) \) by increasing \( T \).

---

**Figure 7:** Optimal debt policy gambling for redemption

when \( B \leq \bar{B}(0;p,0) \) does not bind

---

![Optimal debt policy gambling for redemption](image-url)
In figure 7, we illustrate two possibilities, which depend on \( B_0 \). In one \( T = 1 \), and in the other \( T = 2 \).

The algorithm for calculating the optimal policy on a grid of bond levels is a straightforward application of policy function iteration. Since we work backwards from the period in which the government borrows at price \( \beta p \) and will default in the next period unless a recovery of the private sector occurs, let us reverse our labeling of the value functions and define

\[
V_0(B,0) = \max u((1-\theta)\bar{A}, \theta\bar{A} + \beta p B' - B) \\
+ \beta(1-p) \frac{u((1-\theta)\bar{Z}, \theta\bar{Z})}{1-\beta} + \beta p \frac{u((1-\theta)\bar{Z}, \theta\bar{Z} + (1-\beta)B'))}{1-\beta}
\]

s.t. \( \bar{B}(0; p, 0) \leq B \leq \bar{B}(1; p, 0) \).

The steps of the algorithm are

1. Solve for the value function \( V_0(B,0) \) and the policy function \( B_0'(B) \) on a grid of bonds \( B \) on the interval \( [B, \bar{B}(0; p, 0)] \). We can set the lower limit \( B \) equal to any value, including a negative value. In an application with a given initial stock of debt, we could set \( B = B_0 \). We have already solved this problem analytically. The solution is \( B'(B) = \min[\hat{B}'(B), \bar{B}(1; p, 0)] \) unless \( B'(B) < \bar{B}(0; p, 0) \), in which case \( B_0'(B) = \bar{B}(0; p, 0) \). Consequently,

\[
B_0'(B) = \max\left[ \bar{B}(0; p, 0), \min[\hat{B}'(B), \bar{B}(1; p, 0)] \right].
\]

It turns out that the values of \( B \) for which \( B'(B) < \bar{B}(0; p, 0) \) are those for which it is not optimal to set \( T = 0 \).

2. Let \( t = 0 \), and set \( \hat{B}_0 = \bar{B}(0; p, 0) \).

3. Solve for the value function \( V_{t+1}(B,0) \) and the policy function \( B_{t+1}'(B) \) in Bellman’s equation

\[
V_{t+1}(B,0) = \max u((1-\theta)\bar{A}, \theta\bar{A} + \beta B' - B) + \beta(1-p)V_t(B',0) + \beta p \frac{u((1-\theta)\bar{Z}, \theta\bar{Z} + (1-\beta)B))}{1-\beta}
\]

Let \( \tilde{B}_t \) be the largest value of \( B \) for which \( V_{t+1}(B,0) \geq V_t(B,0) \).
4. Repeat step 3 until $\tilde{B}_t = \overline{B}$.

Let $T$ be such that $\tilde{B}_T = \overline{B}$. We can prove that $\overline{B} < \tilde{B}_{T-1} < \tilde{B}_{T-2} < \cdots < \tilde{B}_1 < \overline{B}(0; p, 0)$. Our algorithm divides the interval $[\overline{B}, \overline{B}(0; p, 0)]$ into subintervals $[\overline{B}, \tilde{B}_{T-1}], [\tilde{B}_{T-1}, \tilde{B}_{T-2}], \ldots, [\tilde{B}_1, \overline{B}(0; p, 0)]$. If the initial capital stock $B_0$ is in the subinterval $[\tilde{B}_1, \tilde{B}_{T-1}]$, then the optimal government policy is to increase $B$, selling debt $B(0; p, 0) < B \leq \overline{B}(1; p, 0)$, in period $t-1$, and defaulting in period $t$ unless the private sector recovers. The optimal sequence of debt is $B_0$, $B_{t-1}'(B_0)$, $B_{t-2}'(B_{t-1}'(B_0))$, $\ldots$, $B_0(B_1(\cdots(B_{t-1}'(B_0))))$.

5. Numerical results for the general model

We now solve numerically the full model in order to evaluate when gambling for redemption might happen in equilibrium.

In order to solve for the equilibrium we need to choose a functional form for the utility function. We choose a log specification, i.e. $u(c, g) = \log(c) + \gamma \log(g - \bar{g})$. The parameters we choose for our benchmark scenario are the following:

<table>
<thead>
<tr>
<th>Table 1: Parameter values in the Benchmark scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>$A$</td>
</tr>
<tr>
<td>$Z$</td>
</tr>
<tr>
<td>$p$</td>
</tr>
<tr>
<td>$\beta$</td>
</tr>
<tr>
<td>$\pi$</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>$\theta$</td>
</tr>
<tr>
<td>$\bar{g}$</td>
</tr>
</tbody>
</table>
Several parameters deserve some comments. We choose a default penalty of 5%, as in Cole and Kehoe (1996). More recently Sosa-Padilla (2014) builds a model where default triggers a banking crises and endogenously generates an output loss due to a sovereign default. His quantitative exercise suggests an output loss in the order of 6%. Our benchmark cost of default is slightly smaller in magnitude, however we assume it is a permanent cost while in his model the output loss would be transitory. We will later perform sensitivity analysis and report results for a larger cost of default.

Regarding the probability of a recovery, we have chosen a parameter that seems reasonable in light of previous recessions. However, it might well be that the probability of a recovery is smaller.

The probability of investors' panic is also arbitrary and we fix it at 3%, generating an interest rate of 5% while in the crisis zone. This magnitude is consistent with the risk premia we have observed in the data (see Figure 1).

Finally, we have set a minimum government expenditure level at 30 percent of initial output. The interpretation of that is an economy that in good times collects 40 percent of output in government revenues, and sees these revenues fall to 36 because of the recession. The minimum expenditure (could be interpreted as entitlements that leave less room for discretionary spending) implies that the curvature of utility is high once the recession hits.

In our theoretical exercise we restricted our analysis to one-period bonds. Clearly, this assumption is very restrictive for the economies in the Eurozone we consider. Table 2 shows the weighted average maturity and the fraction of debt issued at one year or less in the GIIPS and Germany.

<table>
<thead>
<tr>
<th></th>
<th>Weighted average years until maturity</th>
<th>Percent debt with one year or less maturity at issuance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Germany</td>
<td>6.4</td>
<td>5.2</td>
</tr>
<tr>
<td>Greece</td>
<td>15.4</td>
<td>9.8</td>
</tr>
<tr>
<td>Ireland</td>
<td>4.5</td>
<td>0.0</td>
</tr>
<tr>
<td>Italy</td>
<td>6.5</td>
<td>17.0</td>
</tr>
<tr>
<td>Portugal</td>
<td>5.1</td>
<td>14.5</td>
</tr>
<tr>
<td>Spain</td>
<td>5.9</td>
<td>13.2</td>
</tr>
</tbody>
</table>

In order to understand the crucial role of debt structure for our results we extend the model to introduce the feature that a given fraction $\delta$ of the existing stock of debt will mature.
every period. As in Hatchondo and Martinez (2009) or Chaterjee and Eyigungor (2012) this feature renders the maturing of debt “memoryless”, so that it is not necessary to keep track of the entire distribution of debt.

Now the government’s problem is to choose \( c, g, B', z \) to solve

\[
V(s) = \max \ u(c, g) + \beta E[V(s')]
\]

s.t. \( c = (1 - \theta)y(a, z) \)

\[ g + z\delta B = \theta y(a, z) + q(B', s)(B' - (1 - \delta)B). \]

The whole structure of debt would determine the amount of debt due at any given period, \( \delta \), and this is not necessarily related to average maturity. Notice that this problem reduces to the one-period debt case when \( \delta = 1 \), or to infinitely-lived debt as \( \delta \to 0 \).

Accordingly, we need to change the pricing functions. In the benchmark scenario, where \( \overline{b}(0) < \overline{b}(1) < \overline{B}(0) < \overline{B}(1) \), we can define prices recursively (under the scenario where the state of the economy \( s \) implies no default, otherwise prices are zero):

\[
q(B', s) = \begin{cases} 
\beta [\delta + (1 - \delta)q'(\cdot)] & \text{if } B' \leq \overline{b}(0) \\
\beta(p + (1 - p)(1 - \pi)) [\delta + (1 - \delta)q'(\cdot)] & \text{if } \overline{b}(0) < B' \leq \overline{b}(1) \\
\beta(1 - \pi) [\delta + (1 - \delta)q'(\cdot)] & \text{if } \overline{b}(1) < B' \leq \overline{B}(0) \\
\beta p(1 - \pi) [\delta + (1 - \delta)q'(\cdot)] & \text{if } \overline{B}(0) < B' \leq \overline{B}(1) \\
0 & \text{if } \overline{B}(1) < B' 
\end{cases}
\]

Our benchmark scenario is a model where \( \delta = 1/6 \), which is the value implied by a uniform debt structure of six period bonds, consistent with the empirical evidence of Table 2.

In Figure 9 we plot the policy functions, together with the debt thresholds (as described in Figure 8).
First, notice the value of the thresholds. The lower threshold is 60% of GDP, below this level the government would not find it optimal to default even if it couldn’t rollover its stock of debt. Any debt above this level makes the economy vulnerable to a potential panic (given investors beliefs), in which case a crisis would occur and default would be optimal. Second, the upper threshold is close to 105% GDP, above this level interest payments are so large that the government chooses to default even if investors were willing to refinance the stock of debt.

If the economy is in good times then the optimal policy is to keep debt constant when the economy is in the safe region and decrease debt step by step when the economy is in the crisis zone. This is exactly the policy prescription of Cole and Kehoe (2000).

Consider now that the economy unexpectedly shifts to the recession state. Figure 10 plots the impact of such a change.
The first thing to notice is that as soon as the economy enters into the recession state both the lower and upper threshold shift to the left. In our numerical example countries with very low level of debt (above 35% of the original GDP) are still safe, and as a result they do consumption smoothing and increase their level of debt progressively until \( b(0) \).

The region between \( b(0) \) and \( b(1) \) displays very interesting dynamics. Those where economies that were not vulnerable to a panic before. As the recession hits they become vulnerable and interest rates jump up. In terms of debt dynamics this region is split. Countries that are close to the safe threshold of bad times will choose to lower debt in order to avoid paying the premium. However countries with larger level of debt will gamble for redemption until the safe threshold of good times, and stay there waiting for a recovery.

In the region between \( B(1) \) and \( B(0) \) are economies that were vulnerable before the crisis. The interest rate stays the same as before. Some economies with levels of debt above but close to 60% find optimal to reduce their level of debt and wait for a recovery. However, for levels of debt close to 70% the optimal policy implies gambling for redemption all the way to the upper threshold (the maximum amount that investors would lend). Notice that a default cost of
5% is big enough (or the probability of a recovery small enough) to deter these countries for gambling one more period and risking default if a recovery does not happen.

Finally, in the region between $B(0)$ and $B(1)$ we find economies that in good times could refinance their debt (incurring a premium) as long as investors were willing to rollover their debt, but that as soon as the economy hits a recession become insolvent and default directly.

Notice that the model accommodates all types of behavior: economies increasing their levels of debt at high, intermediate and zero risk premia, while other economies choose to reduce their levels of debt in order to reduce their interest payments as in Cole and Kehoe (2000).

The maximum level of sustainable debt is small relative to the magnitudes we are observing in the economies we consider, though. As discussed above, a key parameter in our exercise is the cost of default. Higher costs of default imply that the economy could sustain higher levels of debt. Figure 11 displays the policy function in bad times with a default cost of 10 percent.

**Figure 11: Policy function in bad times with higher default penalty**
The main conclusions are unaffected, but the upper thresholds are significantly larger.
In contrast, lowering the discount factor makes governments less patient and therefore the upper thresholds shift to the left making the amount of sustainable debt smaller.

5.1. The role of the maturity structure of debt

In order to understand the role of debt maturity, we solve the model for different values of $\delta$. In particular, we report in Figure 12 results for $\delta \in \{0.5, 1\}$, which would be the values corresponding to a uniform debt structure of one and two period bonds.

As $\delta$ is made smaller the lower and the upper thresholds diverge.

**Figure 12: Policy function in bad times for selected values of $\delta$**

As $\delta$ becomes progressively higher (shorter maturities) the thresholds shift to the left. In fact, the levels of the thresholds become more in line with the experience of emerging economies that borrow short term, where lower levels of debt are sustainable and rollover crises can emerge for low levels of debt.

In contrast, as $\delta$ becomes smaller (say 20 year bonds) the lower and upper thresholds get closer together (rollover risk disappears) and huge levels of debt could be sustained (larger than 200% GDP).

5.2. The role of recovery expectations: Panglossian governments
We refer to governments that have high expectations about a recovery, or at least higher than the market perceptions, as Panglossian governments. In order to illustrate the consequences of this feature we reevaluate our benchmark economy where we assume that the government has expectations of a recovery of 50 percent, compared to the 20 percent of international lenders.

Figure 13 reports the policy function in bad times (no changes for good times) relative to the benchmark case.

Figure 13: Policy function in bad times. Benchmark and Panglossian government

Notice that an optimistic government (in the sense of believing a recovery to be more likely) would be willing to gamble more.

6. Conclusions

We provide a theory that accounts for governments increasing optimally their levels of debt even in a situation where that makes them more vulnerable to a sovereign debt crisis. The key trade-
off is avoiding the risk premium (by lowering debt) vs the benefits of consumption smoothing (by increasing debt) when faced with a recession of uncertain recovery. If the latter effect dominates we call it “gambling for redemption”, and our model can relate this feature of the equilibrium to the fundamentals of the economy and the maturity structure of debt.

So far we have restricted our analysis to a game between a government and atomistic international investors given the set of fundamentals. We leave two relevant extensions for future research. First, extending our analysis to study this type of game in an environment where the government can engage in costly reforms to change the fundamentals. Second, understanding what are the implications of the existence of a third party in the game, an authority with deep pockets that might consider bailing-out the government if a panic occurs.
References


Appendix: The algorithm for computing an equilibrium in the general model

The algorithm computes the four debt thresholds, the value functions, and the policy functions.

In order to compute these objects we proceed in the following steps:

1. Compute the value function $V(B, a, z_{-1}, \zeta)$ of being in the default state, where $B = 0$ and $z_{-1} = 0$. To simplify notation we denote it $V_d(a)$. Notice that these values are independent of the sunspot $\zeta$, that becomes irrelevant once a default has occurred.

The value function of defaulting in normal times, where $a = 1$, is:

$$V_d(1) = u[(1-\theta)Zy, \theta Zy] + \beta V_d(1),$$

which is just a constant:

$$V_d(1) = \frac{1}{1-\beta} u[(1-\theta)Zy, \theta Zy].$$

Similarly, in a recession, where $a = 0$,

$$V_d(0) = u[(1-\theta)AZy, \theta AZy] + \beta p V_d(1) + \beta (1 - p)V_d(0),$$

which is also a constant:

$$V_d(0) = \frac{1}{1-\beta(1-p)} u[(1-\theta)AZy, \theta AZy] + \frac{\beta p}{(1-\beta)(1-p)(1-\beta)} u[(1-\theta)Zy, \theta Zy].$$

Notice that the value functions become the ones obtained above whenever the state of the economy determines that a self-fulfilling debt crisis happens (or has happened in the past). From now on we describe how to compute the value functions in case of no default, and drop the variable that determines whether a government has defaulted in the past $z_{-1}$ and the sunspot $\zeta$ as arguments of the value functions, i.e. from now on $V(B, a)$ is the value function if default has not happened today or anytime in the past.
2. Guess initial values for the thresholds \( \bar{b}(0), \bar{b}(1), \bar{B}(0), \bar{B}(1) \), where \( \bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1) \), and the associated prices (We can also modify the algorithm to calculate an equilibrium where \( \bar{b}(0) < \bar{B}(0) < \bar{b}(1) < \bar{B}(1) \)).

3. Perform value function iteration on a finite grid of values for debt to compute the value function in normal times, \( a = 1 \).

Guess an initial value function in good times if default has not happened in the past: \( \hat{V}(B,1) \).

Then:

3.1. For values of initial debt \( B \leq \bar{B}(1) \):

The value function is: \( V(B,1) = \max \{ V_1(B,1), V_2(B,1) \} \), where \( V_1(), V_2() \) are the value functions if next period bonds are \( B' \leq \bar{B}(1) \) or \( B' \in (\bar{B}(1), \bar{B}(1)) \), respectively:

\[
V_1(B,1) = \max_{B' \leq \bar{B}(1)} \left\{ u\left[(1-\theta)y, \theta y + \beta B' - B\right] + \beta \hat{V}(B',1) \right\}.
\]

\[
V_2(B,1) = \max_{B' \in (\bar{B}(1), \bar{B}(1))} \left\{ u\left[(1-\theta)y, \theta y + \beta (1-\pi) B' - B\right] + \beta (1-\pi) \hat{V}(B',1) + \beta \pi V_d(1) \right\}.
\]

3.2. For high values of initial debt \( B > \bar{B}(1) \): \( V(B,1) = V_d(1) \).

3.3. If \( \max_{B} \left| V(B,1) - \hat{V}(B,1) \right| > \varepsilon \), then \( \hat{V}(B,1) = V(B,1) \) and go to 3.1. Else, go to 4.

4. Perform value function iteration on a finite grid of values for debt to compute the value function in a recession, \( a = 0 \).

Guess an initial value function if we are in a recession and the government has not defaulted in the past: \( \hat{V}(B,0) \). Remember the value function in normal times, \( V(B,1) \), is already known from Step 3. Then:

4.1. For values of initial debt where \( B \leq \bar{B}(0) \), the value function is:

\[
V(B,0) = \max \{ V_1(B,0), V_2(B,0), V_3(B,0), V_4(B,0) \},
\]

where \( V_1, V_2, V_3, V_4 \) are the associated value functions if next period bonds are in the following four regions, \( B' \leq \bar{b}(0) \), \( B' \in (\bar{b}(0), \bar{b}(1)) \), \( B' \in (\bar{b}(1), \bar{B}(0)) \), \( B' \in (\bar{B}(0), \bar{B}(1)) \), respectively:
\[ V_1(B,0) = \max_{B' \leq B(0) \leq B(1)} \left\{ u\left[ (1-\theta) A_y, \theta A_y + \beta B' - B \right] + \beta p \left( \theta B', 1 \right) + \beta (1-p) \tilde{V}(B',0) \right\} \]

\[ V_2(B,0) = \max_{B' \in [B(0),B(1)]} \left\{ u\left[ (1-\theta) A_y, \theta A_y + \beta \left( p + (1-p)(1-\pi) \right) B' - B \right] + \beta p \left( \theta B', 1 \right) + \beta (1-p) \pi V_d(0) + \beta (1-p)(1-\pi) \tilde{V}(B',0) \right\} \]

\[ V_3(B,0) = \max_{B' \in [B(0),B(1)]} \left\{ u\left[ (1-\theta) A_y, \theta A_y + \beta \left( 1-\pi \right) B' - B \right] + \beta p \pi V_d(1) + \beta p \left( \pi V_d(0) + \beta (1-p)(1-\pi) \tilde{V}(B',0) \right) \right\} \]

\[ V_4(B,0) = \max_{B' \in [B(0),B(1)]} \left\{ u\left[ (1-\theta) A_y, \theta A_y + \beta \left( 1-\pi \right) B' - B \right] + \beta p \pi V_d(1) + \beta p \left( \pi V_d(0) + \beta (1-p)(1-\pi) \tilde{V}(B',0) \right) \right\} \]

4.2. For high values of initial debt \( B > \overline{B}(0) \): \( V(B,0) = V_d(0) \).

4.3. If \( \max_B \left| V(B,0) - \tilde{V}(B,0) \right| < \varepsilon \), then \( \tilde{V}(B,0) = V(B,0) \) and go to 4.1. Else, go to 5.

5. Update the threshold values:

5.1. Choose \( \overline{b}_{\text{new}}(0) \) to be the highest point in the grid of \( B \) for which the following condition is satisfied:

\[ \left\{ u\left[ (1-\theta) A_y, \theta A_y - B \right] + \beta p V(0,1) + \beta (1-p) V(0,0) \right\} \geq V_d(0) \]

5.2. Choose \( \overline{b}_{\text{new}}(1) \) to be the highest point in the debt grid for which the following condition is satisfied:

\[ \left\{ u\left[ (1-\theta) A_y, \theta A_y - B \right] + \beta p V(0,1) \right\} \geq V_d(1) \]

5.3. Choose \( \overline{B}_{\text{new}}(0) \) to be the highest point in the debt grid for which the following condition is satisfied:

\[ V(B,0) \geq \left\{ u\left[ (1-\theta) A_y, \theta A_y + q(B') B'(B,0) \right] + \beta p V_d(1) + \beta (1-p) V_d(0) \right\} \]

where \( q(B') = \beta (1-\pi) \) if \( B'(B,0) \leq \overline{B}(0) \) and \( q(B') = \beta p(1-\pi) \) if \( B'(B,0) > \overline{B}(0) \).

5.4. Choose \( \overline{B}_{\text{new}}(1) \) to be the highest point in the debt grid for which the following condition is satisfied:
\[ V(B,1) \geq \{ u[(1 - \theta) Zy, \theta Z_y + \beta(1 - \pi) B'(B,1)] + \beta V_d(1) \} \]

5.5. If \( |\overline{b}_{\text{new}}(0) - \overline{b}(0)| > \varepsilon \) or \( |\overline{b}_{\text{new}}(1) - \overline{b}(1)| > \varepsilon \) or \( |\overline{B}_{\text{new}}(0) - \overline{B}(0)| > \varepsilon \) or \( |\overline{B}_{\text{new}}(1) - \overline{B}(1)| > \varepsilon \), then
\[
\overline{b}(0) = \overline{b}_{\text{new}}(0), \overline{b}(1) = \overline{b}_{\text{new}}(1), \overline{B}(0) = \overline{B}_{\text{new}}(0), \overline{B}(1) = \overline{B}_{\text{new}}(1)
\]
and go to 3. Else, exit.

Note: Notice that the lower thresholds could be computed directly, since no information about policy functions is required. Hence, the iterative procedure would not be strictly necessary, but we choose to do it that way to be consistent with the computation of the upper thresholds (that do depend on the policy function and hence require an iterative procedure).