Partial Answers to Micro Prelim 2/2002

1. b. A pure strategy for player A is a move in each information set for A. Since there are two moves in each of two information sets, there are $2^2 = 4$ pure strategies for A. The same reasoning shows that there are four pure strategies for B. Denote the moves "stay" and "withdraw" for player i in period 1 by s_i and w_i, $i = A, B$, and denote the same moves for player i in period 2 by S_i and W_i. An example of a pure strategy for player i is (s_i, S_i).

c. The pure strategy in answer b is a subgame perfect Nash equilibrium (SPNE), since neither player can get a higher expected payoff by changing strategy, given the other player's strategy, and the same is true in the subgame at the second period after each player stays. The only other pure strategy Nash equilibrium in the subgame of period 2 is with both players withdrawing, and if these are the moves in period 2, then withdrawing is strictly dominant for each player in period 1. Therefore the only other pure strategy SPNE is $\{(w_A, W_A), (w_B, W_B)\}$.

d. In the outcomes of part c, either both investors withdraw in period 1 or both stay for both periods. One might be tempted to say that rational play should lead to one of these SPNEs. The problem with that is that the strategy for a given player in an SPNE is optimal for that player only under the assumption that the other player plays the strategy *in that equilibrium.* Since there are two equilibria, it is not clear how either player could decide which equilibrium strategy the other player is choosing.

2. a. The condition $h_{m\theta} > 0$ implies that the cost of an additional unit of marketing m is higher if θ is higher. θ might be an index measuring the difficulty the monopoly has communicating with its customers.

b. Marginal revenue is $qP_q + P - (c/e)$, where subscripts denote partial derivatives. This rises with marketing intensity if $qP_{qm} + P_m > 0$.

c. The first order conditions are $qP_q + P - (c/e) = 0$ and $qP_m - h_m = 0$. Implicit differentiation with respect to θ yields

$$(1) \qquad \begin{pmatrix} 2P_q + qP_{qq} & qP_{qm} + P_m \\ qP_{qm} + P_m & qP_{mm} - h_m \end{pmatrix} \begin{pmatrix} \partial q/\partial\theta \\ \partial m/\partial\theta \end{pmatrix} = - \begin{pmatrix} 0 \\ -h_{m\theta} \end{pmatrix}$$

Call the first matrix in this equation A. The second order conditions for the optimization imply that the determinant $|A|$ is nonnegative. So $\partial q/\partial\theta = -h_{m\theta}(qP_{qm}+P_m)/|A| < 0$ and $\partial m/\partial\theta = h_{m\theta}(2P_q+qP_{qq})/|A| < 0$ if $|A| \neq 0$. If $|A| = 0$ then the optimal q and m are not differentiable, but the direction of the effect of changing θ is the same as in the differentiable case. Similarly, implicit differentiation with respect to e yields

$$(2) \qquad \begin{pmatrix} 2P_q + qP_{qq} & qP_{qm} + P_m \\ qP_{qm} + P_m & qP_{mm} - h_m \end{pmatrix} \begin{pmatrix} \partial q/\partial e \\ \partial m/\partial e \end{pmatrix} = - \begin{pmatrix} c/e^2 \\ 0 \end{pmatrix}$$

and $\partial q/\partial e = -(c/e^2)(qP_{mm} - h)/|A| \geq 0$ and $\partial m/\partial e = (c/e^2)(qP_{qm} + P_m)/|A| > 0$ where the optimal q and m are differentiable. In the short run, the optimal q and m are lower if θ is higher, and they are higher (or at least no lower) if e is higher.

d. Optimization with respect to e yields the first order equation $(cq/e^2) - r = 0$ in addition to the equations in part c. By implicit differentiation,

$$(3) \qquad \begin{pmatrix} 2P_q + qP_{qq} & qP_{qm} + P_m & c/e^2 \\ qP_{qm} + P_m & qP_{mm} - h_m & 0 \\ c/e^2 & 0 & -2cq/e^3 \end{pmatrix} \begin{pmatrix} \partial q/\partial\theta \\ \partial m/\partial\theta \\ \partial e/\partial\theta \end{pmatrix} = - \begin{pmatrix} 0 \\ -h_{m\theta} \\ 0 \end{pmatrix}$$

Calling the left-most matrix in this equation B, we have $|B| \leq 0$ by the second order condition for optimization, and $\partial e/\partial\theta = h_{m\theta}(c/e^2)(qP_{qm} + P_m)/|B| < 0$ where this derivative exists. The optimal equipment input is nonincreasing in θ.

e. With higher θ, the long run e is lower (or no higher) than in the short run. The long run q and m are the short run values corresponding to the lower e. So the answer to part c implies that the long run values of q and m are no greater than the short run values.

f. By the envelope theorem, the derivative of the maximal long run profit with respect to r is the derivative of the short run profit with respect to r, with e set at its long run value e^*. This derivative is $-e^* < 0$. This value is smaller in magnitude if θ is higher, so higher θ makes the long run profit less sensitive to changes in the equipment price. The envelope theorem allows us to find the effect on long run profit without finding the effect on e.

3. a. For the Edgeworth box, see the Figure link in the Exam Archive website, or pick up a copy in the department office. b. Ex ante Pareto efficient allocations: $m_2^A = \frac{2}{3}m_1^A$, $0 \leq m_1^A \leq 3$.

c. With actuarially fair insurance each consumer buys full coverage with the same expected consumption as in the initial endowment. The consumption vectors are a and b in the figure, $a = (2, 2), b = (1.5, 1.5)$

d. With "self-insurance," the consumers might agree to outcomes on the chord $\overline{cd}$ in figure.

e. The market equilibrium outcome is point e in the figure. A is worse off relative to insurance, B is better off. Aggregate supply of m in state 2 is relatively low, and equilibrium relative price is high, which favors B.

4. a. The function s^* solves the problem $\max \sum_x (x - s(x))p(x|1)$ with respect to $s(\cdot) \geq 0$ subject to the constraint $\sum_x (s(x) - 1)p(x|1) \geq \sum_x s(x)p(x|0)$ since the agent chooses $e = 1$ under the contract s^*. The corresponding Lagrange function can be written as $L = K + (\mu - 1)\sum_x s(x)p(x|1) - \mu \sum_x s(x)p(x|0)$, where K does not depend on s. The necessary first order conditions imply

$$(\mu - 1)p(x|1) - \mu p(x|0) \leq 0, \text{ with equality if } s^*(x) > 0, \ \forall x. \tag{4}$$

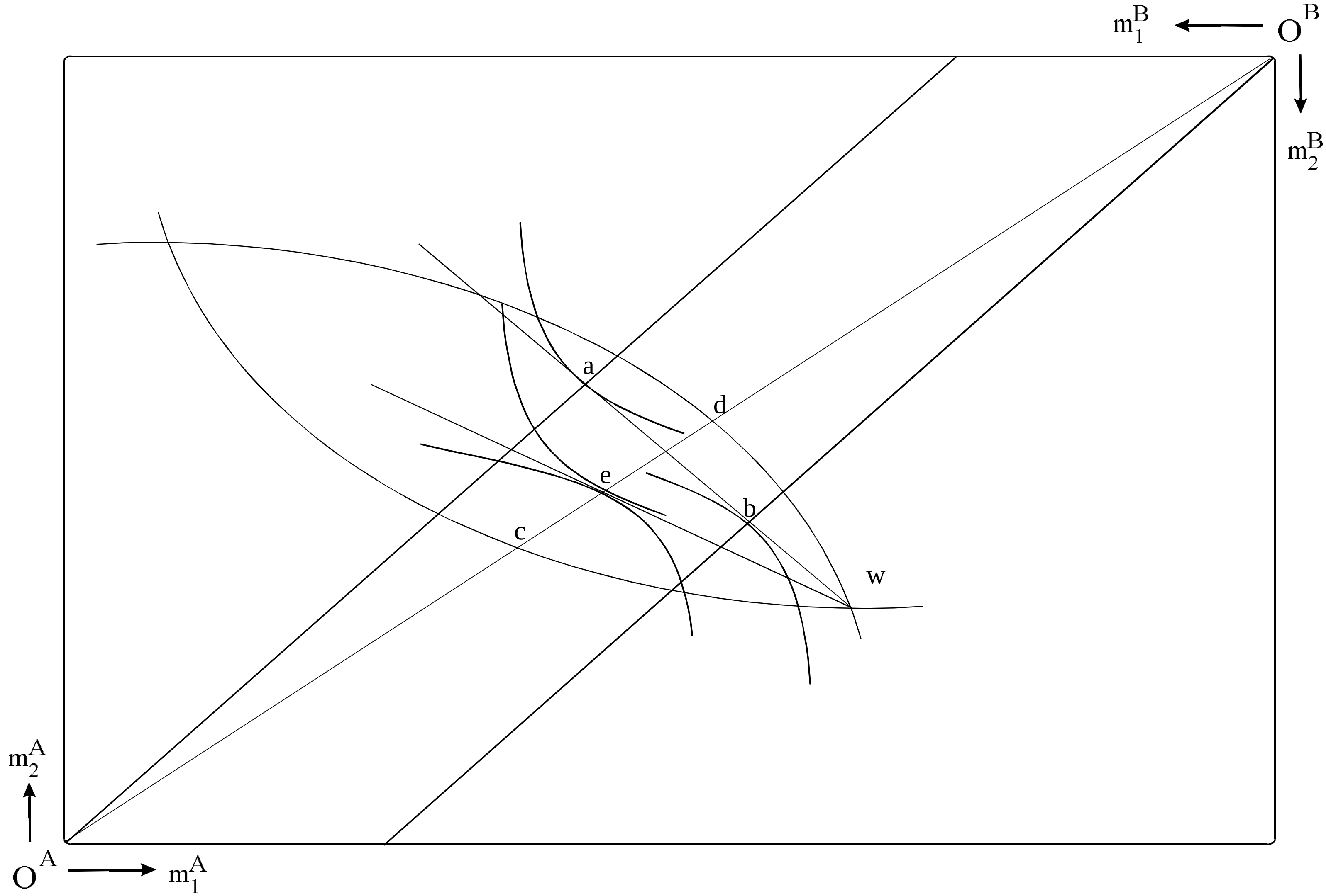
Suppose that $s^*(1) > 0$. Then (4) implies $\mu > 0$ and $(\mu-1)p(x|1)/p(x|0) = \mu > 0$ for $x = 1$. But then (4) is violated for larger x, since $p(x|1)/p(x|0)$ is increasing in x. This shows that $s^*(1) = 0$. The same argument shows that $s^*(2) = 0$. The monotone likelihood property ($p(x|1)/p(x|0)$ increasing) implies that the most efficient way for the principal to induce effort is to give the maximal payment for the highest output and the minimum possible payment for all other outcomes.
b. There are two additional constraints: $s(2)-s(1) \leq 1$ and $s(3)-s(2) \leq 1$. These imply the third constraint $s(3) - s(1) \leq 2$. Suppose that one of the first two (say the first) is violated. Suppose also that the "output" is simply the profit from a transaction. If the profit is 1, then the agent would gain by returning to the principal a profit of 2 and being paid $s(2) > s(1) - 1$. So this constraint would make economic sense if whenever the profit is x, the principal knows that it is at least x but possibly more.
c. Let λ_1 and λ_2 be the Lagrange multipliers corresponding to the two additional constraints in part b. The agent accepts any nonnegative payment function. Suppose that $S(1) > 0$. If $S(2) = 0$ then $\lambda_1 = 0$ and the same argument as in part a leads to a contradiction. A similar argument rules out $S(3) = 0$ if $S(2) > 0$. But if $S(x) > 0$ for each x then S is not optimal for the principal, since reducing S by a small constant raises the principal's payoff, and the resulting contract is still accepted by the agent and satisfies the incentive constraint. Therefore, $S(1) = 0$. d. The necessary first order conditions become

$$(\mu - 1)p(1|1) - \mu p(1|0) + \lambda_1 \leq 0, \quad \text{with equality if } S(1) > 0, \text{ and} \tag{5}$$

$$(\mu - 1)p(2|1) - \mu p(2|0) + \lambda_2 - \lambda_1 \leq 0, \quad \text{with equality if } S(2) > 0, \text{ and} \tag{6}$$

$$(\mu - 1)p(3|1) - \mu p(3|0) - \lambda_2 \leq 0, \quad \text{with equality if } S(3) > 0. \tag{7}$$

With $S(2) > 0$, (6) holds with equality. If $\lambda_2 = 0$, then (6) implies $\mu > 1$. But then (7) is violated since $p(x|1)/p(x|0)$ is increasing. This shows that $\lambda_2 > 0$ and hence $S(3) - S(2) = 1$. The principal gives the maximum incentive for effort by making the payment for raising the output from 2 to 3 as high as possible under the constraints of part b.
5. a. The auction can be viewed as a Bayesian game in which nature picks players' types (their valuations) independently from the uniform distribution on $[0, 1]$, then the players, knowing the structure of the game and their own types, place bids simultaneously. A player wins by placing the higher bid or by being selected randomly if the bids are equal. The winner's payoff is its valuation minus its bid. The other player's payoff is 0. A pure strategy is a function b assigning to each valuation $\theta_i \in [0, 1]$ a bid $b(\theta_i) \in \mathbb{R}_+$.
b. A pure strategy Bayesian Nash equilibrium of the game is a pair of pure strategies $(\bar{b}_1, \bar{b}_2)$ such that for each θ_j, $\bar{b}_j(\theta_j)$ maximizes the expected payoff of player j ($j = 1, 2$) given the strategy of the other player k, i.e., maximizes $(\theta_j - b_j)\text{prob}\{j \text{ wins by bidding } b_j \text{ when } k \text{ uses strategy } \bar{b}_k\}$ with respect to b_j.
c. In a symmetric equilibrium, the players choose the same strategy, denoted b. This function is nondecreasing since the probability that j wins is nondecreasing in j's bid. If the function is constant over some nonempty open interval, then there is a better strategy for each player. A slightly higher bid at a valuation in the interval raises the probability of winning by a discrete amount and raises the player's expected payoff. Suppose that j's valuation is θ. If j places the bid $b(\theta')$ then its probability of winning is the probability that $\theta_k < \theta'$, and its expected payoff is $\theta - b(\theta'))\theta'$. This expected payoff must be maximized when $\theta' = \theta$. (Since b is an optimal strategy, the bid $b(\theta')$ cannot be better than the bid $b(\theta)$.) Since b is increasing, it is differentiable almost everywhere, and for every θ where it is, $\theta - b(\theta) - \theta b'(\theta) = 0$. The solution to this differential equation is $b(\theta) = \theta/2$. (Valuations are uniformly distributed, so the derivative of b is constant.)
d. A player who bids more than its valuation gets negative expected payoff. If the player bids less than the valuation, then a small increase in its bid increases its probability of winning (or leaves it unchanged) without changing the payment due when the player wins. This increases the expected payoff or leaves it unchanged. So a weakly dominant strategy is to bid the valuation.
e.f. The payment in part c is $(1/2)\max\{\theta_1, \theta_2\}$ when player j has valuation θ_j. The expected value of this payment is the integral over the square $[0, 1] \times [0, 1]$, which is 2 times the integral of $\theta_1/2$ taken over the triangle with vertices $(0, 0), (1, 0), (1, 1)$. This is $2\int_0^1 \int_{\theta_2}^1 (\theta_1/2)d\theta_1 d\theta_2 = 1/3$. With valuations θ_j in part d, the payment is $\min\{\theta_1, \theta_2\}$. The expected payment is the integral of this over $[0, 1] \times [0, 1]$, which is $2\int_0^1 \int_0^{\theta_1} \theta_2 d\theta_2 d\theta_1 = 1/3$.



m_1^B
O^B
m_2^B
a
d
e
b
c
w
m_2^A
O^A
m_1^A