June 2003 Micro Prelim Partial Answers

[1] A coalition of n firms ($n > 2$) engages in a joint venture. Firm i contributes input x_i, and the total benefit resulting from the project is $B(x)$, where x denotes the aggregate amount contributed. However, each firm only incurs the cost associated with its own contribution, $C(x_i)$. Suppose $B(0) = 0$, $B' > 0$ and $B'' < 0$, and that $C(0) = 0$, $C' > 0$ and $C'' > 0$. Each firm receives an equal share of the benefit regardless of its contribution.
a. Formulate the decision problem facing firm i and characterize an optimal solution. Assume that the firms act simultaneously and independently.

Firm i maximizes $(1/n)B(x) - C(x_i)$, where $x = \sum_h x_h$. If there is an optimal choice for firm i given $\{x_h\}$, $h \neq i$, then it satisfies $(1/n)B'(x) - C'(x_i) \leq 0$, with equality if $x_i > 0$.

b. State whether the following is True, False or Uncertain. Explain. "Since a firm can reap a benefit from the joint venture without incurring any of the cost, it is optimal to contribute zero."

If $B'(0) - nC'(0) > 0$, then $B'(x) - nC'(0) > 0$ for small $x > 0$, and each firm wants to contribute as long as the other firms contribute sufficiently little. Thus we cannot conclude that a firm's optimal contribution is 0. The statement in quotes is false.

c. Discuss whether or not the structure described above is sufficient to ensure the existence of a best response by firm i to the contributions $x_{-i} = (x_1, ...x_{i-1}, x_{i+1}, ..., x_n)$ of the $n-1$ other firms.

B' is decreasing and nC' is increasing, but it is possible that $B'(x_i) > nC'(x_i), \forall x_i \geq 0$. Then firm i has no best response when its rivals contribute 0. Firm i has a best response no matter what its rivals do if there is some x such that $B'(x) \leq nC'(0)$.

d. Discuss the efficiency of Nash equilibrium outcomes for this game.

An efficient vector of contributions (x_i) maximizes $B(\sum x_i) - \sum C(x_i)$. The first order condition is $B'(x) - C'(x_i) \leq 0$, with equality if $x_i > 0$. By changing to such a vector from any other vector of contributions, it is possible for the gainers to transfer part of their benefits to the losers so that every firm is better off. If $B'(0) \leq C'(0)$, then there is an efficient NE in which each firm chooses 0 contribution. If $B'(0) > C'(0)$, then no NE is efficient. Proof: If no firm contributes in NE, then the outcome is inefficient. If some firm contributes $x_i > 0$ in NE, then $B'(x) = nC'(x_i) > C'(x_i)$, so the first order condition for efficiency is violated.

e. Suppose the firms' contributions affect their share of the benefit, i.e., firm i would receive a share $f_i(x_1, ..., x_n) \geq 0$, where $\Sigma_i f_i(x_1, ..., x_n) = 1$. Discuss conditions on f_i that would ensure the existence of a best response to x_{-i}.

Fix x_{-i} and let $\phi(x_i) \equiv f_i(x_1, ..., x_n)B(\sum_h x_h) - C(x_i)$. There is a best response as long as there is some $\bar{x}$ such that $\phi(\bar{x}) \geq \phi(x_i), \forall x_i > \bar{x}$. (Then the continuous function ϕ has a maximizer on $[, \bar{x}]$.) A sufficient condition for this is that some $\hat{x}$ satisfies $B'(\hat{x} + \sum_{h\neq i} x_h) = C'(\hat{x})$. This follows from the fact that f_i is bounded above.

f. In contrast to the general expressions indicated above, suppose the benefit and cost functions were given explicitly by $B(x) = \Sigma_i x_i$ and $C(x_i) = x_i^2$. Prove that each firm has a dominant strategy, and determine the (dominant strategy) equilibrium payoff for each firm.

If the firms share the benefits equally, then i maximizes $(1/n)\sum x_h - x_i^2$ by choosing $x_i = 1/(2n)$, which does not depend on the contributions of the other firms. Each firm i gets payoff $(1/n)n[1/(2n)] - [1/(2n)^2] = (2n-1)/(4n^2)$.

g. Again assume B and C are as in part f, and assume any subcoalition would face the same benefit and cost schedule on its own. Prove that it would be advantageous for two of the firms to separate from the others and engage in a joint venture of their own. In light of the structure of this economy, discuss why this advantage occurs.

Since $(2n-1)/(4n^2)$ is decreasing in n, two firms do better by separating from the others. As the number of firms grows, the benefit to a particular firm from a given distribution of contributions falls, but the cost remains the same. As a result, the total contribution does not grow in proportion to the number of firms, and the NE net benefit falls.

[2] Consider an economy with a continuum of workers and a continuum of firms. Workers care about their wage earnings but are also concerned about the conditions of their work environment. Specifically, individual i has preferences represented by the utility function

$$u_i(c, e) = c + \delta_i e,$$

where c denotes consumption, e takes on the value 1 if the work environment is pleasant and -1 if it is unpleasant, and δ_i measures the extent to which individual i is concerned about the work environment. Workers are thus characterized by their δ_i. For the entire population of workers, δ is distributed uniformly on $[0, 1]$. Workers have no nonlabor income, and they each supply labor inelastically.

Each firm hires one worker and produces output with value 1. In addition, firms can expend resources to ensure that the working environment is pleasant (i.e., that $e = 1$), but this reduces the value of their output. Firms differ in their cost of providing a pleasant work environment. Specifically, if firm j were to do so, it would reduce the value of its output to $\alpha_j \leq 1$. Firms are thus characterized by their α_j; and among all firms, α is distributed uniformly on $[0, 1]$. If a firm fails to expend any resources, then the environment is unpleasant. Finally, w_1 is the wage paid to a worker in a pleasant environment (measured in units of output or consumption) and w_0 is the wage in an unpleasant environment.

a. What determines each worker's choice of job?

A worker of type δ prefers a pleasant environment if and only if $w_1 + \delta > w_0 - \delta$. This determines the workers' choices if they are free to choose.

b. Given $w_0 - w_1$, which workers choose to work in unpleasant environments?

Workers choose an unpleasant environment if their $\delta < (w_0 - w_1)/2$ and possibly if their $\delta = (w_0 - w_1)/2$.

c. What determines whether a given firm provides a pleasant environment or not?

A firm of type α gets profit $\alpha - w_1$ per employee if its environment is pleasant and and gets profit $1 - w_0$ otherwise if w_1 and w_0 are the wages it must pay in these situations.

d. Given $w_0 - w_1$, which firms invest in worker satisfaction?

Firms invest if their $\alpha > 1 - (w_0 - w_1)$ and possibly if $\alpha = 1 - (w_0 - w_1)$.

e. Find the equilibrium $w_0 - w_1$ and the equilibrium proportion of pleasant jobs.

The fraction of workers choosing unpleasant jobs, $(w_0 - w_1)/2$, equals the fraction of firms offering such jobs, $1 - (w_0 - w_1)$, so $w_0 - w_1 = 2/3$, and $2/3$ of the jobs are pleasant.

f. Which individuals end up in the unpleasant jobs and to which firms are they assigned? How does equilibrium utility vary with δ and how do equilibrium profits vary with α? Explain.

Workers with $\delta \in [0, 1/3)$ and possibly some with $\delta = 1/3$ get unpleasant jobs and work for firms with $\alpha \in [0, 1/3]$. A worker of type δ gets utility $w_0 - \delta$ for $\delta < 1/3$ and utility $w_1 + \delta$

for $\delta > 1/3$. Thus, workers who do not care much about the job environment have utility decreasing in δ and workers who care a lot about it have utility increasing in δ. A worker of type 1/3 is indifferent between the two environments, and so is the worst off type. Workers with lower or higher delta are better off.

Firms of type $\alpha < 1/3$ get profit $1 - w_0$, and firms of type $\alpha > 1/3$ get profit $\alpha - w_1$, which is increasing in α. Thus the firms' profits rise with α if their cost of providing a pleasant environment is low enough. Otherwise, their profit does not vary with α.

[3] Half of the workers in a particular population have high productivity and the rest have low productivity. High and low productivity types with education level e produce respectively $8+2e$ and $1+4e$ units of output per day no matter where they work. Workers in this population choose their (publicly observable and verifiable) education level $e \in [0, 3]$ knowing their type. Then each worker enters a job market in which two firms simultaneously and independently offer wages to the worker. The worker accepts one firm's offer or else rejects both firms and obtains utility 0. If the worker accepts a firm's offer at the wage w then the worker obtains utility $w - e^2$ if its productivity is low and $w - (1/2)e^2$ if its productivity is high. If a firm hires the worker at a wage w then the firm obtains a payoff equal to the worker's productivity minus w. The above conditions are common knowledge among the workers and firms.
a. Suppose first that each worker's type is publicly observable and verifiable. Find all combinations of education and wage for the different types of workers that can arise in subgame perfect Nash equilibrium. Justify your answer.

If a high type with education e is paid more than $8 + 2e$, then the employer can make more profit by offering a lower wage. If the high type is not offered a wage of at least $8 + 2e$ then a firm that employs the worker with probability less than 1 can raise its profit by offering a wage slightly higher than the wage offered by its rival (and higher than $e^2/2$) so that the worker accepts for sure. Thus in subgame perfect Nash equilibrium (SPNE) a high type with education e is paid $8 + 2e$. A high type chooses $e = 2$, which maximize $8 + 2e - (e^2/2)$, and is paid 12. By similar reasoning, a low type chooses $e = 2$, which maximizes $1 + 4e - e^2$, and is paid $1 + 4e = 9$.

b. Evaluate the efficiency of the outcome allocations in part a. Are they Pareto efficient? Explain carefully what this means in the present model.

A feasible allocation in the model of part a can be identified by an education level and output and wage for each worker and an assignment of the worker to a particular firm or to no firm. The allocation determined in part a is Pareto efficient. For the utility of a worker to be increased, the worker must be paid more than its productivity. For this to be feasible, the profit of some firm or the consumption of another worker must be reduced.

NOTE In the rest of the problem, assume that the firms cannot observe a worker's type when they make their wage offers.

c. Find all possible education and wage combinations received by the different worker types in pure perfect Bayesian equilibrium (PBE). Show all your work. Use one or more diagrams to illustrate these education and wage combinations. NOTE that the workers cannot choose e greater than 3.

In pure PBE, the firms both believe that with probability $\mu(e) \in [0, 1]$ a worker with education level e has high productivity. By the same argument as in part a, both firms offer a worker with education e the wage $w(e) = \mu(e)(8+2e)+(1-\mu(e))(1+4e)$, which makes their expected profit 0. Therefore the equilibrium wage offered to a worker with education e is at least $1+4e$, and the low type worker obtains at least the utility 5 in part a. Similar reasoning implies that the high type gets at least the utility 17/2, the maximum of $1 + 4e - (e^2/2)$ subject to $e \leq 3$.

In a separating equilibrium, the different types choose different education levels. Rationality of beliefs implies that a firm, observing an equilibrium education level, assigns probability 1 to the type that chooses that education in equilibrium. From the formula for $w(e)$ above, the types are paid their productivities. Therefore the low type gets utility 5 from $e = 2$ and wage $w = 9$ as in part a. The contract chosen by the high type, $(8 + 2e_H, e_H)$ cannot give the low type utility higher than 5, so $8 + 2e_H - e_H^2 \leq 5$ and $e_H \leq 3$, which implies $e_H = 3$. The high type is paid $8 + 2 \cdot 3 = 14$.

In a pooling equilibrium, the different types choose the same education level, say e. When it is chosen, the firms gain no information about the worker's type. So $\mu(e) = 1/2$ and $w(e) = (9/2) + 3e$. The low type must get utility 5 or more, so $(9/2) + 3e - e^2 \geq 5$, hence $e \leq (3+\sqrt{7})/2 \approx 2.82$. The high type gets at least utility 17/2, so $(9/2)+3e-(e^2/2) \geq 17/2$, hence $e \geq 2$. Every $e \in [2, (3 + \sqrt{7})/2]$ can arise in a pooling equilibrium. Just let $\mu(e') = 0$ for $e' \neq e$. Then for both types, e is at least as good as any other choice of education level.

d. Are some of the equilibrium outcomes from part c more plausible than others? Explain.

Pooling equilibria are less plausible than separating equilibria. Separating equilibria require that when a worker has education level 3, the firms believe the worker's productivity is high, and if the worker has less education, they believe that there is positive probability that the worker has low productivity. This is reasonable since additional education is less costly for the high type. On the other hand, in a pooling equilibrium in which the worker chooses education level e, the firms assign probability below 1/2 to the high type when they observe education $e' > e$. (If $\mu(e') \geq 1/2$ then the high type prefers e' to e since $(9/2) + 3e - (e^2/2)$ is increasing on $[0, 3]$.) Thus observing a higher education level makes the firms attach higher probability to the low type even though the low type never chooses more education than the high type if the two are offered the same wage function.

e. Evaluate the efficiency of the equilibrium outcomes in part c. Are any outcomes Pareto efficient?

f. For each Pareto inefficient equilibrium outcome from part c, determine whether it would be possible for a social planner with the same information as the firms to achieve a Pareto improvement. What can be concluded about the efficiency of screening in a labor market like the one above? Justify your conclusions.

e, f: From parts a and b, only the pooling equilibrium outcome with $e = 2$ is Pareto efficient. Pooling equilibria with $e > 2$ are constrained Pareto inefficient. To see this, note that a planner can achieve a Pareto improvement by offering an additional education and wage combination with a slightly lower education level $e - \epsilon$ and a wage that is lower by 4ϵ. For sufficiently small $\epsilon > 0$, this contract is accepted by the low type (since the marginal cost of education for that type is below 4 at $e > 2$) and is rejected by the high type, since for that type, the marginal cost of education is below 3. The resulting allocation is feasible and raises the utility of the low type without affecting the high type.

The separating equilibrium outcome is Pareto inefficient since the utility of the high type can be raised by assigning it education level 2. But the separating equilibrium is constrained Pareto efficient. Every pooling outcome is worse for the high type. In order to improve on the separating outcome, a planner would have to offer (e_L, w_L) and (e_H, w_H) with $e_L \leq 3$, $e_H \leq 3$, $w_L - e_L^2 \geq 5$, $w_H - (e_H^2/2) \geq 19/2$, $w_L - e_L^2 \geq w_H - e_H^2$, and and at least one inequality strict, with total profit $8 + 2e_H - w_H + 1 + 4e_L - w_L$ nonnegative. This is possible only if the maximum of the total profit subject to the other inequalities is positive. Given the values of

w_H and e_H, total profit is maximized subject to a constraint on the utility of the low type when $e_L = 2$. Also, it is optimal to set w_H as low as possible, i.e., equal to $(19 + e_H^2)/2$. The problem is then to maximize total profit, $17 + 2e_H - [(19 + e_H^2)/2] - w_L$, subject to $e_H \leq 3$, $w_L \geq 9$ and $w_L \geq e_L^2 + w_H - e_H^2 = (27 - e_H^2)/2$. This last inequality implies $w_L \geq 9$, so total profit is maximized when $2e_H - (e_H^2/2) + (e_H^2/2)$ is maximized. This occurs at $e_H = 3$, and in that case, total profit is 0. This shows that no Pareto improvement is possible when the planner has the information available to the firms.

In a screening model, the firms independently offer menus of contracts and the worker accepts a contract or rejects all the contracts offered. By the usual Bertrand argument, both firms get 0 expected profit in SPNE. Suppose that there is a pooling eq in which both worker types accept the same contract. The workers' indifference curves have different slopes at that contract, so it is possible for a firm to raise its expected profit by offering a contract with slightly more education and a slightly higher wage which only the high type accepts, or else slightly less education and a slightly lower wage, which only the low type will accept. This shows that there is no pooling SPNE in the screening model.

The only separating screening equilibrium has the outcome in the separating eq of part c, since it is the only outcome in which each worker type is paid its productivity, gets at least the utility it can get from a wage $1 + 4e$ and weakly prefers the contract that it accepts to the one the other type accepts. This follows from the derivation of the separating eq in part c. The argument above showing that the separating eq is constrained efficient also shows that no firm can increase its expected profit by offering other contracts. The screening equilibrium outcome is Pareto inefficient, but constrained efficient.

g. Suppose that before the workers choose their education levels the firms can simultaneously, independently and credibly commit themselves to offering particular wages for particular education levels. What can be said about subgame perfect Nash equilibrium outcomes of the resulting game played by a particular worker and the two firms? Be as specific as possible with the given information and interpret your conclusions. This question can be answered using the answer to part f above.

The ability to commit to contracts places the firms in the situation of the screening model in which they first make contract offers. So the outcome is the unique separating equilibrium outcome in part c.

[4] Consider the following two period model of a financial contract. In the first period, a lender offers a menu of contracts to a borrower. A contract consists of L (≥ 0) the amount of loan and t (≥ 0), the amount the borrower has to repay in the second period. Assume that the borrower does not default. The borrower either chooses an item in the menu or rejects the offer. If a contract (L, t) is offered and accepted, the lender receives utility $t - rL$, where r is the riskless alternative lending interest rate while the borrower invests the loan and produces $\theta f(L)$, obtaining utility $\theta f(L) - t$. Assume that $f(0) = 0$ and for L strictly positive, $f'(L) > 0$ and $f''(L) < 0$. Also, for any $r > 0$, there is $L > 0$ such that $f'(L) = r$. The productivity parameter θ can take value θ_h with probability p and value θ_l with probability $1 - p$, where $\theta_h > \theta_l > 0$ and $1 > p > 0$. The borrower learns her type θ before she obtains the loan. The borrower's reservation utility is zero.

a. Describe the contracts lender would offer if he can identify the type of borrower before offering a contract. Explain why a contract offered by the lender and satisfying the participation constraint of the borrower yields non-negative utility for the lender.

For type θ, the optimal contract for the lender solves $\max t - rL$ subject to $\theta f(L) - t \geq 0$. $\mathcal{L} = t - rL + \lambda(\theta f(L) - t)$. $\mathcal{L}_t = 1 - \lambda = 0$, $\mathcal{L}_L = -r + \lambda\theta f'(L) = 0$. Thus, $\theta f'(L) = r$. Since

$\theta_h f'(L_h) = r = \theta_l f'(L_l)$, $L_h > L_l$. $t_h = \theta_h f(L_h) > t_l = \theta_l f(L_l)$. Recall $(L,t) = (0,0)$ is a feasible contract. Thus, $\theta_h f(L_h) - rL_h \geq 0 - r0 = 0$. Similarly, $\theta_l f(L_l) - rL_l \geq 0$.

b. Now, suppose that the lender does not learn the type of borrower when he makes the contract offer. Formulate the problem of the lender when he wants to maximize his expected utility subject to incentive and participation constraints. In the formulation, let (L_h, t_h) be a contract aimed at type θ_h and (L_l, t_l) be a contract aimed at type θ_l.

$\max p\,(t_h - rL_h) + (1-p)\,(t_l - rL_l)$ subject to
$\theta_h f(L_h) - t_h \geq \theta_h f(L_l) - t_l$
$\theta_l f(L_l) - t_l \geq \theta_l f(L_h) - t_h$
$\theta_h f(L_h) - t_h \geq 0$
$\theta_l f(L_l) - t_l \geq 0$.

c. Show that at an optimal contract for the lender, the participation constraint of the low productivity borrower is binding.

Suppose at the optimum, $\theta_l f(L_l) - t_l > 0$. Since by definition an optimum satisfies all constraints, $\theta_h f(L_h) - t_h \geq \theta_h f(L_l) - t_l > \theta_l f(L_l) - t_l > 0$. Thus, increasing both t_h and t_l by a small amount would increase the utility of the lender without violating any constraints.

d. Show that at an optimal contract for the lender, the incentive constraint of the high productivity borrower is binding.

Suppose $\theta_h f(L_h) - t_h > \theta_h f(L_l) - t_l$. Then, $\theta_h f(L_h) - t_h > \theta_h f(L_l) - t_l > \theta_l f(L_l) - t_l \geq 0$. Then, just increasing t_h by a small amount would increase the utility of the lender without violating any constraints.

e. Show that the participation constraint of the high productivity borrower and the incentive constraint of the low productivity borrower are satisfied so long as the binding constraints in c and d are satisfied and $L_h > L_l$.

Suppose $\theta_h f(L_h) - t_h = \theta_h f(L_l) - t_l$ and $\theta_l f(L_l) - t_l = 0$. Then, $\theta_h f(L_h) - t_h = \theta_h f(L_l) - t_l > \theta_l f(L_l) - t_l = 0$. If $L_h > L_l, \theta_l\,(f(L_h) - f(L_l)) \leq \theta_h\,(f(L_h) - f(L_l)) = t_h - t_l$. From this, $\theta_l f(L_l) - t_l \geq \theta_l f(L_h) - t_h$.

f. Solve the problem of the lender and interpret the solution. Show in your solution, $L_h > L_l$. Compare the solution with solutions in a. (you can assume that $L_l > 0$ at the optimum).

Let $\mathcal{L} = p\,(t_h - rL_h) + (1-p)\,(t_l - rL_l)$ subject to $\theta_h f(L_h) - t_h = \theta_h f(L_l) - t_l$ and $\theta_l f(L_l) - t_l = 0$. Making substitutions, $\mathcal{L} = p\,(\theta_h f(L_h) - \theta_h f(L_l) + \theta_l f(L_l) - rL_h) + (1-p)\,(\theta_l f(L_l) - rL_l)$ $= p\,(\theta_h f(L_h) - \theta_h f(L_l) - rL_h) + (1-p)\,(-rL_l) + \theta_l f(L_l)$. Thus, $\mathcal{L}_{L_h} = p\,(\theta_h f'(L_h) - r) = 0$. $\mathcal{L}_{L_l} = p\,(-\theta_h f'(L_l)) + (1-p)\,(-r) + \theta_l f'(L_l) = 0$, the latter assuming that $L_l > 0$ at the optimum. $\theta_h f'(L_h) = r$ and $\theta_l f'(L_l) = p\theta_h f'(L_l) + (1-p)\,r = p\theta_h f'(L_l) + (1-p)\,\theta_h f'(L_h)$ $= \theta_h\,(pf'(L_l) + (1-p)\,f'(L_h))$. $\frac{\theta_l}{\theta_h} f'(L_l) = pf'(L_l) + (1-p)\,f'(L_h)$. Since $\frac{\theta_l}{\theta_h} < 1$, $pf'(L_l) + (1-p)\,f'(L_h) < f'(L_l)$. This implies $f'(L_h) < f'(L_l)$,or $L_h > L_l$. From e. all the constraints are satisfied. The equation $\theta_h f'(L_h) = r$ shows that for the high productivity type, the first best choice of loan is obtained although $t_h = \theta_h f(L_h) - (\theta_h - \theta_l) f(L_l) < \theta_h f(L_h)$. Since 'zero contracts' are available, the lender's utility is non-negative.