Partial Answers to Microeconomics Preliminary Examination of September 8, 2004

1a. For each of the three moves in the information set T_U player 1 has three moves in the information set T_D, so player 1 has $3^2 = 9$ pure strategies.
b. The payoff to player 1 after U is higher than either payoff after R, so all three pure strategies (R, λ), (R, δ) and (R, ρ) for player 1 are strictly dominated. Every other pure strategy for either player is a best response to some strategy of the player's rival, so there are no other strictly dominated pure strategies.

c. If player 1 ends the game with U its payoff is higher than with δ, so assume that T_U is reached if the research project is successful. The choices U and δ by firm 1 lead to no new interaction with firm 2. A move to the left (ℓ) or right (r) by firm 1 could represent the choice of a product to develop that is related to outputs of firm 2. The players agree in their preference ranking over moves of player 2 at each node where player 2 moves. The choices F and f could represent investment in research and choices N and n could represent investment in marketing by firm 2.

One possibility is that product ℓ is complementary with an output of firm 2 whereas product r is a closer substitute of an output of firm 2. If firm 1 chooses ℓ after successful research, then additional research by firm 2 is not so productive and marketing by firm 2 is more beneficial for both firms (N is preferred to F). If firm 1 chooses ℓ after unsuccessful research, then research by firm 2 can improve the combination of the firms' products and is preferred by both firms (F preferred to N after λ). If firm 1 chooses r, then marketing (n) by firm 2 gives it a higher payoff than research if firm 1's research was unsuccessful. The marketing might also help firm 1 by increasing awareness of that type of good. If firm 1's research was successful, then research by firm 2 is more productive and is preferred by both firms (maybe because producers of closer substitutes can learn rivals' research results more easily). This research spillover can account for firm 1 getting a higher payoff after R, f than after R, n. The cost of research must be included in the payoffs of player 1 and successful research must be more expensive than unsuccessful research in order for firm 1 to have higher payoff after ρ, n than after R, n.

d. Let μ_ℓ and μ_r be probabilities player 2 attaches to player 1 having moved from T_U when player 2 is in its left or right information set, respectively. In WPBE with player 1 choosing L and ρ, $\mu_\ell = 1$ and $\mu_r = 0$. Sequential rationality requires that player 2 chooses N and n, and L and ρ are best responses for player 1.

e. If player 1 chooses U and λ in WPBE, then the ℓ information set of player 2 is reached with positive probability, and $\mu_\ell = 0$ in the notation of part d. Therefore, player 2 chooses F in information set ℓ. Let σ_j be the probability that a player chooses move j in the information set where that move can be chosen. Sequential rationality requires player 1's payoff 2 from λ to be at least as great as its expected payoff $3\sigma_n + 1 - \sigma_n$ from ρ, so $\sigma_n \leq 1/2$. Player 2 plays f with positive probability, therefore its expected payoff of $2\mu_r$ from f is at least as great as its expected payoff of $1 - \mu_r$ from n, with equality if $\sigma_n > 0$. Thus, $\mu_r \geq 1/3$, with equality if $\sigma_n > 0$. WPBE imposes no other restrictions, so in WPBE with $\sigma_U = 1$ and $\sigma_\lambda = 1$, we have $\sigma_F = 1$, $\mu_\lambda = 0$, and either $\mu_r \in [1/3, 1]$ and $\sigma_n = 0$ or else $\mu_r = 1/3$ and $\sigma_n \in (0, 1/2]$.

f. In all the WPBEs of part e, player 2 in information set r attaches positive probability to being at the upper node. But playing R with positive probability is strictly dominated for player 1, so this belief is unreasonable. If player 2 attaches 0 probability to being at the upper node ($\mu_r = 0$) then it chooses n with probability 1 and λ is not a best response for player 1.

Under the interpretation of part c, in the WPBE of part e, after firm 1 chooses to develop the closer substitute, firm 2 believes that firm 1's research might have been successful. But with successful research, firm 1 would have preferred not to develop the substitute. If firm 2 believes this, then it concludes that firm 1's research was unsuccessful if it develops the substitute. In that case, firm 2 invests in marketing rather than research since it will not benefit from research results of firm 1.

2. Consider a competitive firm which manufactures output y using labor L and an intermediate good x according to the technology $y = f(x, L)$. It can choose either of two options: (1) it can *outsource* the entire production of x and buy it competitively at the price p_x, or (2) it can produce its entire demand for x *in-house* from additional labor, using the technology $x = g(L)$.
a. First, assuming the firm were to outsource x, write the full decision problem it faces and characterize a solution. What conditions on f would ensure the existence of a solution? (Try to make them as weak as possible.)

The firm maximizes $p_y f(x, L) - p_x x - wL$, where w is the wage rate. If f is continuous and bounded, then the maximization problem has a solution no matter what p_x and w are, as long as they are positive. Even if f is unbounded (but is still continuous), the problem has a solution if there is a compact subset of $\mathbb{R}^2$ such that for (x, L) outside the set, the partial derivatives of f satisfy $f_1(x, L) \leq p_x$ and $f_2(x, L) \leq w$. In that case, there is a profit maximizer in the compact set and no point outside the set yields more profit. Note that this condition depends on p_x and w, not on f alone. If f is differentiable at a solution (x, L), then $p_y f_1(x, L) \leq p_x$, with equality if $x > 0$, and $p_y f_2(x, L) \leq w$, with equality if $L > 0$.

b. Suppose the firm were to produce x in-house. Write the decision problem it faces in this case and characterize a solution. Now discuss conditions on f and g that would ensure the existence of a solution.

The firm maximizes $p_y f(g(\ell), L) - w(\ell + L)$. A solution exists if the function $F(\ell, L) \equiv f(g(\ell), L)$ is continuous and bounded or if it is continuous and there is a compact set such that the partial derivatives F_1 and F_2 are no greater than w outside the set. If f and g are differentiable at a solution (ℓ, L), then $p_y f_1(g(\ell), L)g'(\ell) \leq w$, with equality if $\ell > 0$, and $f_2(g(\ell), L) \leq w$, with equality if $L > 0$.

c. Next, assume the firm was not constrained to outsource all of x or produce it all in-house, but could do some of each. Characterize the optimal production plan for the firm.

The firm chooses x, ℓ and L to maximize $p_y f(g(\ell) + x, L) - p_x x - w(\ell + L)$. Assuming that f and g are differentiable, at a solution (x, ℓ, L), $p_y f_1(g(\ell) + x, L) \leq p_x$, with equality if $x > 0$; $p_y f_1(g(\ell) + x, L)g'(\ell) \leq w$, with equality if $\ell > 0$; and $p_y f_2(g(\ell) + x, L) \leq w$, with equality if $L > 0$.

e. Now suppose the firm could ensure the quality of the x it produced in-house, but not if it were outsourced. However, it is known that 95% of outsourced units prove to be satisfactory while 5% are defective and cannot be used. Assuming the firm must either outsource the entire production or produce all units in-house, characterize its optimal decision.

If the defective outsourced units of x are not paid for, then the optimization problem with outsourcing is the same as in part a, with x interpreted as the number of satisfactory units. To get x units the firm orders $x/(.95)$ units. If the defective units *are* paid for, then x satisfactory units cost $p_x x/(.95)$, which is $p_x/(.95)$ per unit. The firm's problem is to maximize $p_y f(x, L) - (p_x x/.95) - wL$, with first order condition $p_y f_1(x, L) \leq p_x/(.95)$

with equality if $x > 0$, and $p_y f_2(x, L) \leq w$ with equality if $L > 0$. The optimization with in-house production is the same as in b. The firm solves each problem and chooses the production method that yields the most profit. The in-house cost of x units of intermediate good is $wg^{-1}(x)$, so in-house production is preferred to outsourcing if the $\max\{p_y f(x, L) - (p_x x/.95) - wL\} < p_y f(x^*, L^*) - wg^{-1}(x^*) - wL^*$ for some x^* and L^*. This requires $wg^{-1}(x^*) < p_x x^*/(.95)$, so that the cost of in-house production is less than the cost of outsourcing when x^* units of intermediate good are used.

f. Finally, suppose the firm were a monopolist in its output market but still purchased labor competitively and, in the event it outsourced, faced a competitive market for x. Could this affect its decision whether to outsource or produce in-house? Explain.

Yes. Define the cost functions $c(q, p_x, w) = \min\{(p_x x/.95) + wL : f(x, L) \geq q\}$ with outsourcing and $C(q, w) = \min\{wg^{-1}(x) + wL : f(x, L) \geq q\}$ with insourcing. If the firm is a monopolist and is forced to outsource, then its output q_m is less than if it acted competitively and outsourced. It is possible that the in-house cost $C(q_m, w)$ is less than the outsoucing cost $c(q_m, p_x, w)$ even though $C(q_c, w) > c(q_c, p_x, w)$ (the in-house cost is greater than the outsourcing cost at the output level q_c that is optimal if the firm acts competitively. The reverse can also be true, so that it is optimal for the firm to outsource if it is a monopoly and produce in-house if it acts competitively. The latter case is more likely if the firm has significant increasing returns in the production of the intermediate good. Then $g^{-1}(x)$ rises rapidly with x for small x and less rapidly for larger x, so that there is more chance that in-house production is cheaper than outsourcing at larger scale production levels.

3. Consider an exchange economy with two agents, 1 and 2, and two commodities, x and y. Agent 1's preferences are represented by the utility function $u_1(x, y) = xy$ and 2's preferences are represented by $u_2(x, y) = -(x - 1)^2 - (y - 1)^2$. The agents' endowments are given by $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$, respectively.

a. Draw an indifference map depicting agent 2's preferences.
b. Identify the set of Pareto efficient allocations.
c. Determine agent 2's Walrasian demand functions for x and y.
d. Does there exist a competitive equilibrium in this economy? If so, find it. If not, explain why not.
e. Discuss the validity of the first and second welfare theorems in this environment.
f. In general, discuss any problems, either concerning individual market behavior or the aggregate market outcome, that may arise if some agents have utility functions such as agent 2's.

a. Each indifference curve of agent 2 is part of a circle centered at $(1, 1)$ (the part contained in the Edgeworth box with a total of 2 units of each good.)

b. The allocation $((1, 1), (1, 1))$ is Pareto efficient since it is preferred by agent 2 to every other allocation. If any other interior allocation $(x_i, y_i)_i$ is efficient then $x_2 \leq 1$ and $y_2 \leq 1$ (otherwise reducing one of these consumption levels raises the utility of both consumers) and $(1 - x_2)/(1 - y_2) = y_1/x_1 = (2 - y_2)/(2 - x_2)$ (the agents have equal marginal rates of substitution). Therefore $(1 - x_2)(2 - x_2) = (1 - y_2)(2 - y_2)$ and $x_2 = y_2$ since $(1 - x)(2 - x)$ is decreasing in x on $[0, 1)$. This implies $x_1 = y_1$ so the set of efficient allocations is the set of pairs $((2 - x, 2 - x), (x, x))$ with $x \in [0, 1]$.

c. If at prices (p_x, p_y) and wealth w, agent 2 can afford $(x, y) = (1, 1)$ then that is what it chooses. This occurs if $p_x + p_y \leq w$. Suppose that this last inequality does not hold. If agent 2's demand violates the budget identity $p_x x + p_y y = w$, then there is an affordable consumption vector closer to $(1, 1)$ that agent 2 prefers. Therefore the budget identity is satisfied. Agent 2 consumes $(x, y) \gg 0$, if its marginal rate of substitution equals the price ratio: $(1 - x)/(1 - y) = p_x/p_y$ and $p_x x + p_y y = w$, hence $p_y(1 - x) = p_x - p_x y$, $y = (p_x - p_y + p_y x)/p_x$, $p_x x + (p_x - p_y + p_y x)(p_y/p_x) = w$ and $x = (wp_x - p_x p_y + p_y^2)/(p_x^2 + p_y^2)$ and, by symmetry, $y = (wp_y - p_x p_y + p_x^2)/(p_x^2 + p_y^2)$. If the expression for x is negative, then the optimal x is 0 and the optimal y is w/p_y. Similarly, if the expression for y is negative, then the optimal y is 0 and the optimal x is w/p_x.

d. The economy is symmetric with respect to the goods. (If the labels x and y are interchanged there is the economy remains the same.) This suggests that there might be a competitive eq in which the goods are treated the same, i.e., have the same price. If $p_x = p_y$, then each consumer chooses consumption vector $(1, 1)$, and the allocation is feasible, so this is a competitive eq.

e. The fundamental welfare theorems do not apply to this economy since consumer 2 does not have locally nonsatiated preferences. But the conclusions of the theorems are still correct. Each competitive eq with transfers is Pareto efficient and every Pareto efficient allocation is part of a price quasiequilibrium (in fact a price equilibrium) with transfers. In all cases, the equilibrium price of x equals the equilibrium price of y.

f. In general, the conclusions of the fundamental welfare theorems need not hold if a consumer's preferences do not satisfy local nonsatiation. In addition, competitive eq might not exist. This occurs in the economy above if the endowment distribution is changed so that consumer 2 is sufficiently wealthy. For example, if the endowments of x and y are changed to $(1, 1/2)$ for consumer 1 and $(1, 3/2)$ for consumer 2, then consumer 1 has no optimal consumption if either price is nonpositive. If both prices are positive, then consumer 2 chooses $(1, 1)$ in the interior of its budget set, and there is excess supply of at least one good with a positive price, hence no competitive eq.

4. Consider a private ownership economy described as follows. Each firm j, $j = 1, \ldots, J$ ($J > 1$) has a production set $Y_j = \{y_j = (y_{1j}, \ldots, y_{Lj}) \in \mathbb{R}^L : F^j(y_j) \leq 0, y_{1j} \geq 0, y_{jj} \geq 0, y_{kj} \leq 0,\ \text{for}\ k \neq 1, k \neq j\}$, where F^j is continuously differentiable with $F^j(0) = 0$ and where y_{kj} is the firm's net output of good k, $k = 1, \ldots, L$. Each consumer i, $i = 1, \ldots, I$ ($I > 1$) has a continuously differentiable utility function $u_i : \mathbb{R}_+^L \to \mathbb{R}$, an endowment vector $e_i = (e_{1i}, \ldots, e_{Li}) \in \mathbb{R}_+^L$ and a profit share vector $(\theta_{ij})_j \in \mathbb{R}_+^J$, where θ_{ij} is the fraction of the profit of firm j received by consumer i. Define any notation you introduce.

a. The inequalities $y_{1j} \geq 0$, $y_{jj} \geq 0$ and $y_{kj} \leq 0$ in the definition of Y_j imply that firm j cannot be a net user of goods 1 and j as inputs and cannot be a net producer of any other good.

b. If the function F^j is replaced by $2F^j$, there is no effect on Y_j. The set of y_j with $F^j(y_j) \leq 0$ is the same as the set of y_j with $2F^j(y_j) \leq 0$.

c. Use the notation above to give a formal definition of a Pareto efficient allocation (also called Pareto optimal) for this economy.

An allocation $\bar{a} = ((\bar{x}_i)_{i=1}^I, (\bar{y}_j)_{j=1}^J)$ is Pareto efficient in this economy if it is feasible (i.e., if $\bar{x}_i \geq 0, \forall i$, $\bar{y}_j \in Y_j, \forall j$ and $\sum_i(\bar{x}_i - e_i) = \sum_j \bar{y}_j$) and there is no feasible allocation $((x_i)_{i=1}^I, (y_j)_{j=1}^J)$ such that $u_i(x_i) \geq u_i(\bar{x}_i), \forall i$ with strict inequality for some i.

d. Let $\bar{a} = ((\bar{x}_i)_{i=1}^I, (\bar{y}_j)_{j=1}^J)$ be a Pareto efficient allocation for the economy. Let $\bar{u}_i \equiv u_h i(\bar{x}_i), \forall i$. For each consumer h, $\bar{a}$ must maximize $u_h(x_h)$ over the set of feasible allocations

$((x_i)_{i=1}^I, (y_j)_{j=1}^J)$ satisfying $u_i(x_i) \geq \bar{u}_i, \forall i \neq h$. The Lagrange function can be written as $\sum_i \lambda_i[u_i(x_i) - \bar{u}_i] - \gamma[\sum_i(x_i - e_i) - \sum_j y_j] - \sum_j \mu_j F^j(y_j)$, where $\gamma \in \mathbb{R}^L$ and $\lambda_h = 1$. Taking into consideration the constraints on the y_j's, the corresponding first order conditions are $\lambda_i \partial u_i(\bar{x}_i)/\partial x_{\ell i} \leq \gamma_\ell$, with equality if $\bar{x}_{\ell i} > 0$, $\gamma_\ell - \mu_j \partial F^j(\bar{y}_j)/\partial y_{\ell j} \leq 0$ for $\ell = 1$ or j, with equality if $\bar{y}_{\ell j} > 0$, and $\gamma_\ell - \mu_j \partial F^j(\bar{y}_j)/\partial y_{\ell j} \geq 0$ for all other ℓ, with equality if $\bar{y}_{\ell j} < 0$; and also $\mu_j F^j(\bar{y}_j) = 0, \forall j$, and $\gamma[\sum_i(x_i - e_i) - \sum_j y_j] = 0$.

If a strictly quasiconcave function is maximized over a convex set, there is at most one solution. The first order conditions found above are the same as for the problem $\max \sum_i \lambda_i u_i(x_i) - \sum_j \mu_j F^j(y_j)$ subject to $\sum_i(x_i - e_i) - \sum_j y_j = 0$. If each u_i is strictly quasiconcave and each F^j is strictly quasiconvex, then there is a unique solution for the x_i's and y_j's with $\lambda_i > 0$ and $\mu_j > 0$. Suppose that $\partial u_h(\bar{x}_h) \gg 0$ and every u_i is locally nonsatiated in a neighborhood of 0. Then the first order conditions imply $\gamma \gg 0$, hence $\mu_j > 0, \forall j$. For any i with $\lambda_i = 0$, the first order conditions imply $\bar{x}_i = 0$, so there is a unique solution to the original maximization problem.

e. In this part of the problem only, assume that in the Pareto efficient allocation $\bar{a}$ above, $\bar{x}_i \gg 0$ for every consumer i. State conditions on the functions u_i and F^j (as weak as possible) ensuring that $\bar{a}$ is part of a price equilibrium with transfers, i.e., that there is a price vector p and a wealth level w_i for each consumer i such that $\bar{x}_i$ maximizes $u_i(x_i)$ subject to $px_i \leq w_i, \forall i$ and $\bar{y}_j$ maximizes py_j over $y_j \in Y_j$, $j = 1, \ldots, J$. Explain, with the help of diagrams, why the conditions you have stated cannot be dispensed with. Show how a price vector p in such a price equilibrium with transfers can be computed if you know $\bar{a}$ and the functions $F^j, \forall j$ and $u_i, \forall i$. Is it necessary to know all of these functions or would it be enough to know just one of them? Explain.

Sufficient conditions are that each u_i is quasiconcave and locally nonsatiated, and each F^j is quasiconvex. This makes all preferences and production sets convex. MW Figures 16.D.6, 16.D.3 and 15.C.3(a) show that the conclusion need not hold if these conditions are violated. A price vector in a price eq with transfers is $\partial u_1(\bar{x}_1)$, which must be nonnegative since $\bar{x}_1 \gg 0$. Comparing the first order conditions in part d to the first order conditions satisfied at a solution to the consumer's optimization in a price eq with transfers shows that this price vector or scalar multiples of it are the only ones possible in price equilibria with transfers.

f. Suppose now that the functions u_i and F^j are all concave for all i, j and that the allocation $\bar{a} = ((\bar{x}_i)_{i=1}^I, (\bar{y}_j)_{j=1}^J)$ is **NOT** Pareto efficient. Is it necessarily true that there are two agents (two firms or two consumers or a consumer and a firm) that can benefit simultaneously by exchanging just two goods between themselves starting from their production or consumption vectors in the allocation $\bar{a}$? (The other agents in the economy remain with their consumption or production in allocation $\bar{a}$.) Show that your answer is correct. You may use your answer to part d.

There might not be any pair of agents who can benefit by exchanging two goods with each other. For example, if no production is possible and if each consumer i has a unit of good i and no other endowment, and each $i < L$ wants good $i + 1$ and no other good (consumer L wants good 1), then the initial allocation is Pareto inefficient. Giving the endowment of i to consumer $i - 1$ for each $i > 1$, with consumer 1's endowment going to consumer L yields a Pareto improvement. But it is not possible for two agents to benefit simultaneously from trading with each other.